WAVE EQUATION IN HIGHER DIMENSIONS-EXAMPLES.

Kirchhoff's formula. Consider the initial-value problem for the wave equation in $\mathbb{R}^3$:

$$u_t - \Delta u = 0, u = u(x,t), x \in \mathbb{R}^3, t > 0 \quad u(x,0) = u_0(x), u_t(x,0) = u_1(x).$$

The solution is:

$$u(x,t) = t\times \text{ average value of } u_1 \text{ over } S_t(x) + \frac{d}{dt}[t\times \text{ average value of } u_0 \text{ over } S_t(x)].$$

Here $S_t(x) = \{y \in \mathbb{R}^3; |y - x| = t\}$ is the surface of the sphere with center x, radius t. This may also be written as:

$$u(x,t) = \frac{1}{4\pi t}\int_{S_t(x)} u_1(y)dA(y) + \frac{d}{dt}[\frac{1}{4\pi t}\int_{S_t(x)} u_0(y)dA(y)].$$

The proof is based on the *method of spherical averages*: any radial solution $u(r,t)$ of the wave equation in $\mathbb{R}^3$ $(r = |x|)$ has the property that the function $v(r,t) = ru(r,t)$ is a solution of the one-dimension wave equation $v_{tt} - v_{rr} = 0$ on the half-line $\{r > 0\}$ with Dirichlet boundary condition $v(0,t) = 0$.

Example 1. Solve the initial-value problem for the heat equation in $\mathbb{R}^3$ $(c = 1)$, with initial data $u(x,0) \equiv 0, u_t(x,0) = |x|^2 = r^2$.

Answer. $u(r,t) = r^2 t + t^3$. (to check this, use the fact that on radial functions in $\mathbb{R}^3$: $\Delta u = u_{rr} + \frac{2}{r}u_r$.)

Example 2. Solve the wave equation in $\mathbb{R}^3$ (with $c = 1$), for the initial data $u_0 \equiv 0$, u_1 the characteristic function of $B_R = \{y \in \mathbb{R}^3; |y| \leq R\}$, the ball with radius R centered at the origin.

By Kirchhoff's formula, the solution is:

$$u(x,t) = \frac{1}{4\pi t} \times \text{Area of}(S_t(x) \cap B_R).$$

This area can be computed by integration in spherical coordinates (using also the Law of Cosines: done on class.) We find (with $|x| = r$):

$$A = \frac{t\pi}{r}[R^2 - (t-r)^2], \text{ if } |R-r| \leq t \leq R+r, \quad A = 0 \text{ for } t \geq R+r \text{ or } t \leq r-R (\text{ if } r \geq R),$$

and if $R \geq r$ also $A = 4\pi t^2$ for $t \leq R - r$.

By Kirchhoff's formula, the solution is:

$$u(r,t) = \frac{1}{4r}[R^2 - (t-r)^2], |R-r| \leq t \leq R+r, \quad u(r,t) = 0 \text{ if } t \geq R+r.$$

If $r \geq R$ (that is, x is outside of B_R), we have $u(r,t) = 0$ for $t \leq r - R$ as well. But if $r \leq R$, we have $u(r,t) = t$ for $t \leq R - r$.

Spacetime diagram. It is useful to sketch, in the $r > 0, t > 0$ quadrant, the regions where the solution is given by various expressions (*done in class*). Inside the strip bounded by the parallel lines $t - r = -R, t - r = R$ and the line segment $t + r = R, 0 \leq r \leq R$, the solution is given by the quadratic expression; outside, it is given by 0 or t.

Exercise 1. Verify that the solution $u(r,t)$ is continuous in the closed first quadrant $r \geq 0, t \geq 0$, except for a jump discontinuity at the origin when $t = R$. It is enough to consider the limits at each of the lines in the spacetime diagram separating (or bounding) the regions above. *In addition*, verify that the solution satisfies the initial condition.

Exercise 2. Set $R = 1$ in Example 2. (i) Sketch a graph of the solution $u(\frac{1}{2}, t)$ as a function of $t \geq 0$ (this is the solution as seen by an observer at $x \in \mathbb{R}^3$, if $|x| = \frac{1}{2}$.)
(ii) Sketch a graph of the solution $u(r,t)$ as a function of $r > 0$, for the values $t = 1/2, t = 1, t = 2$ (still for $R = 1$). Note in the graph the facts that (a) for $0 \leq t \leq R$, the solution is constant in the ball (centered at the origin) of radius $R - t$, and has equal to the ball of radius $R + t$; (b) for $t \geq R$, the support of the solution is the annular region $x \in \mathbb{R}^3; t - R \leq |x| \leq t + R$.

Huygens's principle: Initial data at a point $x_0 \in \mathbb{R}^3$ affect the solution only at points *on the surface* of the 'light cone' issuing from $(x_0, 0)$ in spacetime. (Information propagates exactly at the speed of light: no fater and no slower.) As a consequence, if the initial data has support contained in a ball, for large enough t the solution will be supported in an annular region (as above.)

Exercise 3. Now use the above solution to solve the problem with initial conditions: $u_0 =$ the characteristic function of B_R, $u_1 \equiv 0$. Draw the corresponding spacetime diagram, indicate where the solution is discontinuous and answer questions (i) and (ii) of Exercise 2 for this solution.

One-dimensional case, revisited. D'Alembert's formula for the solution in one dimension may be put in a form similar to the above:

$$u(x,t) = u(x,t) = t \times \text{ average value of } u_1 \text{ over } I_t(x) + \frac{d}{dt}[t \times \text{ average value of } u_0 \text{ over } I_t(x)],$$

where $I_t(x)$ is the interval $[x-t, x+t]$.

Exercise 4. (i) Explain why d'Alembert's formula can be put in this form, and answer the question: is Huygens's principle valid in dimension one?

(ii) The solution of the one-dimensional WE $u_{tt} - u_{xx} = 0$ with initial data $u_0 \equiv 0, u_1 =$ the characteristic function of the interval $I_R = [-R, R]$ is given by:

$$u(x,t) = \frac{1}{2} \times Length(I_R \cap I_t(x)), \quad I_t(x) = [x-t, x+t].$$

Compute this length as a function of $r = |x|, R$ and t (as in Example 2), and use this to write down the solution in a form similar to Example 2.

(iii) Sketch a spacetime diagram (in the positive (r,t) quadrant) for this one-dimensional problam, similar to that drawn in class for the three-dimensional case. (Show where the solution takes the value 0, and where different expressions for the solution ae valid.)

(iv) Repeat Exercise 2 for this one-dimensional case.

Solution in two dimensions. ('method of descent'.)

The general idea is to think of a solution of a PDE in n space dimensions as a solution of the same PDE in $n+1$ dimensions ($u(x,y,t)$ where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$) which happens to be independent of the coordinate y. This works for the heat and wave equations, since:

$$\Delta u = \Delta_x u + u_{yy},$$

where by $\Delta_x u$ we mean the Laplacian in $\mathbb{R}^n$, acting on the vector coordinate x.

For instance, consider the wave equation in $\mathbb{R}^2$: $u_{tt} - \Delta u = 0$, with initial data u_0, u_1 in $\mathbb{R}^2$. Kirchhoff's formula expresses the solution at (x,t) in terms of averages of the initial data over spheres in $\mathbb{R}^3$ (with center x, radius t). Since the solution is independent of the third coordinate, it is enough to consider averages over the upper hemisphere. Pursuing this idea (and expressing the upper hemisphere as the graph of a function over the disk $D_t(x)$ in $\mathbb{R}^2$ of center x, radius t) we arrive at the solution formula:

$$u(x,t) = \frac{1}{2\pi} \int_{D_t(x)} \frac{u_1(y)}{\sqrt{t^2 - |y-x|^2}} dA(y) + \frac{d}{dt}[\frac{1}{2\pi} \int_{D_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} dA(y)],$$

where $D_t(x) = \{y \in \mathbb{R}^2; |y-x| \leq t\}$, the disk with center x, radius t.

Note that, since the denominator is zero when $|y - x| = t$, this is an improper integral; but it is easy to show it is convergent (if u_0, u_1 are continuous.)

Also, the integrals are carried out over the entire disk, not just its boundary. This means if the initial data are supported on a small disk D_R in the plane, the solution at time t would be supported on a larger disk (the union of all the disks with center x_0, radius t, for $x_0 \in D_R$ (see Exercise 5 below.) Thus, *Huygens's principle is not valid for the wave equation in two dimensions.*

Exercise 5. Consider the analogue of Exercise 3 in two dimensions. The solution is given by:

$$u(x,t) = \int_{D_t(x)\cap D_R} \frac{dA(y)}{\sqrt{t^2 - |y-x|^2}},$$

where $D_R = \{|y| \leq R\}$ and $D_t(x) = \{|y - x| \leq t\}$ are disks in $\mathbb{R}^2$. y symmetry, this depends only on $r = |x|$.

(i) Find the regions in the (r,t) first quadrant where the solution is zero (spacetime diagram);

(ii) Find the value of the solution at the origin: compute $u(0,t)$ explicitly, and sketch its graph.

Exercise 6. 'Descend' from two to one dimension: show how the one-dimensional d'Alembert formula (for $u_0 \equiv 0$) follows by considering two-dimensional solutions $u(x,y,t)$ of the two-dimensional WE, which happen to be independent of y.

(Some problems based on [Strauss. Ch. 9])