CONNECTEDNESS-Notes

Def. A topological space X is *disconnected* if it admits a non-trivial splitting:

$$X = A \cup B, \quad A \cap B = \emptyset, \quad A, B \text{ open in } X, \text{ and non-empty.}$$

(We'll abbreviate 'disjoint union' of two subsets A and B –meaning $A \cap B = \emptyset$– by $A \sqcup B$.)

X is *connected* if no such splitting exists. A subset $C \subset X$ is connected if it is a connected topological space, when endowed with the induced topology (the open subsets of C are the intersections of C with open subsets of X.)

Note that X is connected if and only if the only subsets of X that are simultaneously open and closed are $\emptyset$ and X.

Example. The real line R is connected. The connected subsets of R are exactly the intervals. (See [Fleming] for a proof.)

Def. A topological space X is *path-connected* if any two points $p, q \in X$ can be joined by a continuous path in X, the image of a continuous map $\gamma : [0,1] \to X$, $\gamma(0) = p, \gamma(1) = q$.

Path-connected implies connected: If $X = A \sqcup B$ is a non-trivial splitting, taking $p \in A, q \in B$ and a path γ in X from p to q would lead to a non-trivial splitting $[0,1] = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$ (by continuity of γ), contradicting the connectedness of $[0,1]$.

Example: Any normed vector space is path-connected (connect two vectors v, w by the line segment $t \mapsto tw + (1-t)v, t \in [0,1]$), hence connected.

The continuous image of a connected space is connected: Let $f : X \to Y$ be continuous, with X connected. If $f(X) = A \sqcup B$ is a nontrivial splitting (with A, B relatively open in $f(X)$: $A = f(X) \cap U, B = f(X) \cap V$, with U, V open in Y). Then (since $f^{-1}(A) = f^{-1}(U)$ and $f^{-1}(B) = f^{-1}(V)$ are open in X, by continuity), $f^{-1}(f(X)) = f^{-1}(A) \sqcup f^{-1}(B)$ is a nontrivial splitting of X (since $f^{-1}(f(X)) = X$, the domain of f being X.) This would contradict the connectedness of X.

An easy corollary is the *general intermediate value theorem:* if $f : X \to R$ is a continuous function from a connected space X and it takes on the values $a, b \in R$ (with $a < b$), then it also takes on any value in the interval $[a, b]$. (That is, $f^{-1}(\{t\}) \neq \emptyset$, for all $t \in [a, b]$).

If $X \subset Y$ is a dense subset, and X is connected, then so is Y. Recall 'dense' means that any non-empty open subset U of Y contains points of X. Equivalently $\bar{X} = Y$, the closure of X in Y is all of Y.

To see this, consider (as usual) a nontrivial splitting $Y = A \sqcup B$, with A, B open in Y. Then $X = (X \cap A) \sqcup (X \cap B)$ would be a nontrivial splitting of X. Note that $X \cap A \neq \emptyset$ if $A \neq \emptyset$: take $a \in A$, and an open neighborhood U of A in Y, with $U \subset A$. Then we may find $x \in U \cap X$, in particular $x \in A \cap X$. (And likewise for $B \cap X$.) This would contradict the connectedness of X.

More generally, if $X \subset Y \subset \bar{X}$ and X is connected, then so is Y. ($Y \subset \bar{X}$ means any *non-empty* open set in Y intersects X.) To see this consider a splitting $Y = A \sqcup B$, with A, B open in Y. Then

$$X = X \cap Y = (X \cap A) \sqcup (X \cap B)$$

is a splitting of X. Since X is connected, one of $X \cap A$ or $X \cap B$ must be empty, and since $Y \subset \bar{X}$ this means A or B is empty, so the supposed splitting of Y is trivial.

Example of a connected subset of R^2 that isn't path connected. Consider the graph and the vertical segment:

$$\Gamma = \{(x, y) | x > 0, y = \cos(1/x)\}, \quad \Sigma = \{(x, y) | x = 0, y \in [-1, 1]\}.$$

The claim is that the disjoint union $C = \Gamma \sqcup \{0\}$ is a connected, but not path-connected subset of the plane. It follows that $\bar{\Gamma} = \Gamma \sqcup \Sigma$ isn't path-connected either.

Note that Γ is dense in $\Gamma \sqcup \Sigma$ ($\bar{\Gamma} = \Gamma \sqcup \Sigma$), and the graph Γ is connected (indeed, homeomorphic to R). So we know $\Gamma \sqcup \Sigma$ is connected; but this does not imply its subset C is connected. On the other hand since $\Gamma \subset C \subset \bar{\Gamma}$, it does follow (as just seen) that C is connected.

To see C is not path-connected, proceed by contradiction. If C were path-connected, in particular there would exist a continuous path $\gamma : [0, 1] \to C$ joining the origin 0 to the point $(\frac{1}{2\pi}, 1) \in \Gamma$, with $\gamma(0) = 0$.

The idea of the proof is that for t in a small interval containing 0, $\gamma(t)$ would have to be close to the origin, while its first component $\gamma^1(t)$ would map that interval to a small interval on the positive x-axis containing 0; however in any such interval the graph Γ has "large swings".

Formally, consider the set $J = \{t \in [0, 1] | \gamma(t) = 0\}$. Clearly J is closed in $[0, 1]$ and non-empty ($0 \in J$). If we show J is open, by connectedness of $[0, 1]$ it would follow that $J = [0, 1]$, contradicting $\gamma(1) = (\frac{1}{2\pi}, 1)$.

Let $t_0 \in J$. There exists an open interval $I_\delta(t_0) = (t_0 - \delta, t_0 + \delta)$ (or $[0, \delta)$, if $t_0 = 0$) contained in $[0, 1]$ so that $||\gamma(t)|| < 1$ for $t \in I_\delta(t_0)$ (by continuity of γ). The image $\gamma^1(I_\delta(t_0))$ of this interval under the first component γ^1 is an interval on the x-axis containing 0. (Connected maps to connected). *We claim this interval is degenerate* (I.e., it is just the trivial interval $\{0\}$.)

Indeed any non-degenerate interval (on the positive x-axis) containing 0 necessarily contains a point of the form $x = \frac{1}{2\pi n}$ (for some large integer $n \geq 1$), so we have $\gamma(t) = (\frac{1}{2\pi n}, 1)$ for some $t \in I_\delta(t_0)$, contradicting $||\gamma(t)|| < 1$.

Thus $\gamma^1(I_\delta(t_0)) = \{0\}$, and since the only point of C with first coordinate 0 is the origin, it follows that γ maps the interval $I_\delta(t_0)$ to the origin, so this interval is contained in J, proving J is open.

Remark: Note that this example shows that the closure of a path-connected set is not necessarily path-connected!

However, in one important case connected implies path-connected.

Proposition. A connected *open* subset of R^n is path-connected. Indeed, any two points may be joined by a polygonal line.

Proof (outline). Let $U \subset R^n$ be open and connected, $p \in U$. Consider the sets $A = \{q \in U | q$ may be joined to p by a polygonal line in $U\}$ and $B = \{q \in U | q$ may not be joined to p by a polygonal line in $U\}$. Using convexity of open balls in R^n, it is not hard to show that both A and B are open in U! Since U can't have a non-trivial splitting, it follows that $B = \emptyset$ (since $p \in A$.)

It is very plausible that a union of connected sets with a point in common is connected, and not hard to prove. (But note that the intersection of two connected sets with a point in common is not always connected!)

Proposition. Let $(X_\lambda)_{\lambda \in L}$ be a family of connected subsets of X with a point in common: $a \in \cap_{\lambda \in L} X_\lambda$. Then $Y = \cup_{\lambda \in L} X_\lambda$ is connected.

Proof. Suppose we had a nontrivial splitting $Y = A \sqcup B$. Say $a \in A$; then $\forall \lambda \in L$, $a \in A \cap X_\lambda$. But for each λ we have $X_\lambda = (X_\lambda \cap A) \sqcup (X_\lambda \cap B)$, where we can't have both sets non-empty, since X_λ is connected. Thus $X_\lambda \cap B = \emptyset$ for all $\lambda \in L$, and hence $B = \cup_{\lambda \in L}(X_\lambda \cap B)$ is empty, and the splitting must be trivial. This shows A is connected.

Corollary 1. X is connected if, and only if, given any two points $a, b \in X$ there exists a connected subset $X_{ab} \subset X$ containing both a and b.

One direction is clear. Now assuming this property, we have for each

$a \in X$: $X = \cup_{b \in X} X_{ab}$, with X_{ab} connected. Since the X_{ab} (for varying b) have the common point a, the conclusion follows from the proposition.

Corollary 2. The cartesian product $X = X_1 \times X_2$ of two spaces (with the product topology) is connected, if and only if, both factors X_1 and X_2 are.

The projections $p_i : X \to X_i$ are continuous, so X_1 and X_2 are connected if X is.

Conversely, assume X_1 and X_2 are connected. Fix $a = (a_1, a_2) \in X$, and consider, for each $x = (x_1, x_2) \in X$ the 'cross':

$$C_x = (X_1 \times \{a_2\}) \cup (\{x_1\} \times X_2).$$

This is connected, as the union of two connected sets with the common point (x_1, a_2). The sets C_x have the common point $a = (a_1, a_2)$, and $X = \cup_{x \in X} C_x$.

Connectedness as a homeomorphism invariant.

If X and Y are homeomorphic, they are both connected or both disconnected. This allows one to use connectedness as a tool to show two given spaces *are not* homeomorphic. Some examples:

1) R and $R^n, n > 1$. (Remove one point.)
2) The unit circle S^1 and any subset of R (Remove one point.)
3) Intervals of the forms (a, b) and $[c, d)$ (Does *any* point disconnect?)

Connected components. Let $x \in X$. The *connected component* of x in X is the union C_x of all connected subsets of X containing x. As seen earlier, C_x is connected (and is the largest connected subset of X containing x.)

If $x, y \in X$, their connected components C_x, C_y are either disjoint, or coincide. Indeed if $z \in C_x \cap C_y$, then $C_x \cup C_y$ is connected, hence coincides with C_x and with C_y. The sets C_x (for some $x \in X$) are the *connected components* of X. Since the closure of a connected set is connected, each C_x must be closed in X. (But see Problem 7 below.)

PROBLEMS

1. Let $E \subset R^n$ be a vector subspace. The complement $R^n \setminus E$ is connected if, and only if, $dim(E) \leq n - 2$.

Hint. Let $d = dim(E)$. Pick a basis for R^n of the form $\{e_1, \ldots, e_d\} \cup \{f_1, \ldots f_{n-d}\})$, where the first set is a basis of E. Letting F be the subspace spanned by the second set we have $R^n = E \oplus F$, and the coordinates of a point in R^n in this basis may be written as $p = (x, y) = \sum_{i=1}^{d} x^i e_i + \sum_{j=1}^{n-d} y^j f_j$, with $x = (x^i) \in R^d, y = (y^j) \in R^{n-d}$. To show two points

$p \neq q$ in $R^n \setminus E$ may be joined by a curve, suppose first $p, q \in F \setminus \{0\}$, or in coordinates $p = (0, y_1), q = (0, y_2)$, $y_1, y_2 \in R^{n-d} \setminus \{0\}$. Normalizing we obtain $\hat{y}_1, \hat{y}_2$ in S^{n-d-1}, the unit sphere in R^{n-d}. Note that since $n-d-1 \geq 1$ these two points may be joined by a curve on the sphere (see the hint to Problem 8). Show this implies y_1, y_2 may be joined by a curve in $R^{n-d} \setminus \{0\}$, then deal with the general case of $p, q \in R^n \setminus E$.

2. If X is path-connected and $f : X \to R^n$ is continuous, $f(X)$ is path-connected.

3. The union of path-connected subsets of R^n with a common point is path-connected.

4. $X_1 \times X_2$ is path-connected if, and only if, both X_1 and X_2 are.

5. (a) The set of invertible $n \times n$ matrices is disconnected (and open) in R^{n^2}.

(b) The set of orthogonal $n \times n$ matrices is also disconnected.

(c) The set of orthogonal $n \times n$ matrices with determinant one is connected (and so is the set of invertible $n \times n$ matrices with positive determinant.)

6. Let $Q \subset R^2$ be a countable set. Then $X = R^2 \setminus Q$ is connected.

Outline. Let $x, y \in X$, and pick any line L in R^2 cutting the segment $[x, y]$ in its interior. For each $z \in L$, we have a 'broken segment' $S_z = [x, z]V[z, y]$. Note they intersect only at the endpoints. One of these must miss Q; if all the S_z encountered Q we would have an injective map from L to Q, which is impossible since L is uncountable.

7. Let $U \subset R^n$ be an open set. Show the connected components of U are open in R^n.

8. Show that S^1 is not homeomorphic to S^n for $n > 1$. (Recall S^n denotes the unit sphere in R^{n+1}.)

Hint. Show first that, for any n, $S^n \setminus \{N\}$ is homeomorphic to R^n (where $N = (0, 0, \ldots, 0, 1) \in R^{n+1}$). Denoting points in R^{n+1} by (y, x_{n+1}), with $y \in R^n$ and $|y|^2 + x_{n+1}^2 = 1$ if $(y, x_{n+1}) \in S^n$, the homeomorphism is:

$$z = f(y, x_{n+1}) = \frac{y}{1 - x_{n+1}} \in R^n.$$

(This map is called *stereographic projection* from N.) This map is clearly continuous. To show it is a homeomorphism, compute the inverse map from R^n to $S^n \setminus \{N\}$ explicitly, and observe it is continuous.

Ans. The inverse map is $z \mapsto (\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}), z \in R^n$.

Remark: Note that this homeomorphism implies S^n and $S^n \setminus \{P\}$ are connected if $n \geq 1$ (for any $P \in S^n$.)