Problems (First homework set)

1. Consider the spaces of infinite sequences of real numbers:

$$\{ v = (x_1, x_2, x_3, \dots) \,;\; x_i \in \mathbb{R} \}$$

$$\ell^\infty = \{ v \mid (\exists M > 0)\ |x_j| \le M \ \forall j \ge 1 \} \quad \text{with} \quad \|v\|_{sup} = \sup\{|x_j| \,;\, j \ge 1\}$$

$$\ell^2 = \{ v \mid \sum_{j=1}^{\infty} |x_j|^2 < \infty \} \quad \text{with} \quad \|v\|_{\ell^2} = \Big(\sum_{j=1}^{\infty} |x_j|^2 \Big)^{1/2}$$

$$\ell^1 = \{ v \mid \sum_{j=1}^{\infty} |x_j| < \infty \} \quad \text{with} \quad \|v\|_{\ell^1} = \sum_{j=1}^{\infty} |x_j|$$

Show the inclusions (a) $\ell^1 \subset \ell^2$ with $\|v\|_{\ell^2} \le \|v\|_{\ell^1}$ $\forall v$

(b) $\ell^2 \subset \ell^\infty$ with $\|v\|_{\ell^\infty} \le \|v\|_{\ell^2}$ $\forall v$.

Hint: note that, for each $N \ge 1$:

$$\cancel{\sup}\ \max\{ |x_j| \,;\, 1 \le j \le N \} \le \Big(\sum_{j=1}^{N} |x_j|^2 \Big)^{1/2} \le \sum_{j=1}^{N} |x_j| .$$

2. (i) Complete the details in Example 5 (a sequence $f_n \in C[0,1]$ which is Cauchy for the L^1 norm, but does not converge in L^1 norm to any $f \in C[0,1]$.)

(ii) Show that $C[0,1]$ is not complete for the L^2 norm $\|f\|_{L^2} = \Big(\int_0^1 |f(t)|^2 dt \Big)^{1/2}$

(is the same sequence Cauchy in L^2?)

[Recall $\int_0^1 |f|(t)\,dt \le \Big(\int_0^1 |f|^2 dt \Big)^{1/2}$ by the Cauchy-Schwarz ineq.] not needed

Hint $\int_0^1 |f-g|^2 dx \le 2M \int_0^1 |f-g|\, dx$ if $\|f\|_{sup} \le M$, $\|g\|_{sup} \le M$ in $[0,1]$.

3. Give an example of a bounded sequence in $(C[0,1], \|\cdot\|_{sup})$ which has no convergent subsequence.

Hint: Examples seen in class when discussing different types of convergence.