MATH 447, FALL 2017-HW set 3

1. *Problems on Lipschitz and Hölder conditions.*

(i) Show that $f(x) = \sqrt{x}, x \in [0, \infty)$ is *NOT* Lipschitz continuous on any interval of the form $[0, L]$ $(L > 0)$.

(ii) Show that the same function is Hölder continuous on $[0, \infty)$, with exponent 1/2.

(iii) Suppose a function $f : R \to R$ satisfies, for some $M > 0$ and some $\theta > 1$:

$$|f(x) - f(y)| \leq M|x - y|^{\theta}, \quad \forall x, y \in R.$$

Show that f is constant on R. (*Hint:* Prove that f is differentiable on R, with $f'(x) = 0$ everywhere.)

(iv) For a Lipschitz continuous, bounded function $f : A \to R$, $A \subset R^n$, define the 'best Lipschitz constant' by:

$$[f]_{Lip} = \sup\{\frac{||f(x) - f(y)||}{||x - y||}; x, y \in A, x \neq y\}.$$

Although this is not a norm, adding the sup norm does define a norm in the vector space $Lip_b(A)$:

$$||f||_{Lip} = ||f||_{sup} + [f]_{Lip}.$$

Show that this norm is complete. (You will need part (v).)

(v) Let $f_n : A \to R$ $(A \subset R^m, n \geq 1)$ be a sequence of continuous functions on A which is uniformly Lipschitz, meaning $[f_n]_{Lip} \leq M$, with M independent of n. Suppose $f_n \to f$ uniformly on A. Then f is Lipschitz continuous on A, and $M + 1$ is a Lipschitz constant for f.

Briefly: a uniform limit of uniformly Lipschitz functions is Lipschitz. (And similarly for Hölder continuous functions.)

2. *Problems on uniform continuity.*

(i) If $f : R^n \to R^m$ is continuous, f is uniformly continuous on any bounded subset $A \subset R^n$.

(ii) $f(x) = \cos x$ is uniformly continuous on $\mathbb{R}$, but $g(x) = \cos(x^2)$ is not.

(iii) A polynomial $p : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$ if, and only if, it has degree 0 or 1.

(iv) If $f : I \to \mathbb{R}$ is continuous, monotone and bounded (where $I \subset \mathbb{R}$ is an interval), then f is uniformly continuous on I. (*Hint:* If f is continuous on I and $J, K \subset I$ are intervals with an endpoint in common and such that $I = J \cup K$, then f is uniformly continuous on I if it is uniformly continuous separately on J and K (*show this.*)

(iv bis) *Monotone functions.* If $f : I \to R$ is monotone (where $I \subset R$ is an interval) all discontinuities of f in I are of 'jump type' (one-sided limits exist and are finite); thus the set of discontinuity points is at most countable.

(v) If $f_n : A \to \mathbb{R}$ are uniformly continuous on $A \subset R^n$ and $f_n \to f$ uniformly on A, then f is uniformly continuous on A.

(vi) No sequence of polynomials may converge uniformly to the function $\frac{1}{x}$, or to the function $\sin(\frac{1}{x})$, on the open interval $(0, 1)$. (In contrast, any continuous function in the closed interval $[0, 1]$ may be approximated by polynomials, uniformly in $[0, 1]$.)

(vii) Find an example of $f : R \to R$ differentiable on R, uniformly continuous on R, with unbounded derivative (or prove it cannot exist.) *Hint:* Consider $f(x) = x^2 \sin \frac{1}{x^2}$ in $[-1, 1]$, extended linearly to $\{|x| \geq 1\}$.

3. The goal of this problem is to prove that each open subset $A \subset R$ of the real line can be written as a countable disjoint union of open intervals:

$$A = \bigcup_{j \geq 1} (a_j, b_j), \quad (a_i, b_i) \cap (a_j, b_j) = \emptyset \text{ for } i \neq j.$$

(We allow one of the b_j to be $+\infty$, one of the a_j to be $-\infty$.)

(i) Define the equivalence relation on A: for $x, y \in A$

$$x \sim y \Leftrightarrow [x, y] \subset A$$

($[x, y]$ is the closed interval from x to y, or from y to x, taken to be $\{x\}$ if $x = y$.) Show each equivalence class is an open interval contained in A, *maximal* in the sense that it is not contained in any larger interval contained in A. (Obviously, equivalence classes are disjoint.)

(ii) Define an injective map from the set of equivalence classes to the rational numbers, using the intervals found in part (i). (Recall the rational numbers are dense in R). This shows there are only countably many equivalence classes.

4. *Definition:* A continuous map $f : R^n \to R^n$ is *open* if $f(U)$ is open in R^n, whenever U is open in R^n.

(i) Show that a continuous, injective, open map $f : R^n \to R^n$ is a homeomorphism from R^n to the open set $V = f(R^n) \subset R^n$.

(ii) Show that a strictly monotone continuous function $f : R \to R$ defines a homeomorphism from R to the open set $V = f(R)$. (Strictly monotone means $x < y \Rightarrow f(x) < f(y)$.)

5. *Problems on the convex hull.*

(i) Let $H \subset R^n$ be an $(n-1)$-dimensional subspace, and $x \in R^n$ be a point not in H. Denote by $x * H$ the union of all line segments from x to points of H. Then $C(\{x\} \cup H) = x * H$. Also:

$$x * H = \{y \in R^n; 0 \leq y \cdot x < ||x||^2\} \cup \{x\},$$

provided we choose the origin so that $H = \{y \in R^n; y \cdot x = 0\}$.

(ii) If $A \subset R^n$ is closed, $C(A)$ is not necessarily closed.

(iii) If $A \subset R^n$ is compact, $C(A)$ is compact.

(iv) If $A \subset R^n$ is open, $C(A)$ is open.

6. *Problems on distance functions.*

For two sets $A, B \subset R^n$, and a given point $x \in R^n$, we define:

$$d(x, A) = \inf\{||x-y||; y \in A\}, \quad d(A, B) = \inf\{||x-y||; x \in A, y \in B\} \text{ (euclidean norm).}$$

We have:

$d(x, A) = d(x, \bar{A})$ and $d(A, B) = d(\bar{A}, \bar{B})$. $d(x, A) = 0$ if and only if $x \in \bar{A}$.

(i) It is possible for two closed sets A, B to be disjoint, with $d(A, B) = 0$ (give an example.)

We have the inequality: $d(x, A) \leq d(z, A) + ||x - z||$. This implies:

$$|d(x, A) - d(z, A)| \leq ||x - z||, \forall x, z \in R^n.$$

Thus the distance function $x \mapsto d(x, A)$ is Lipschitz in R^n, and 1 is a Lipschitz constant.

(ii) If A is closed, given $x \in R^n$ there exists a $y \in A$ so that $d(x, A) = ||x - y||$. Such a y is in general not unique (example?), but it is unique if A is (closed and) *convex*.

(iii) If A is compact and B is closed (with $A \cap B = \emptyset$), we may find $x \in A$, $y \in B$ so that $d(A, B) = ||x - y||$. (In particular, $d(A, B) > 0$.) But this is false if we just assume A is closed.

(iv) Let $A, B \subset R^n$ be closed and disjoint. Define $f : R^n \to [0, 1]$ via:

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Then f is continuous, takes the value 0 exactly on A, and the value 1 exactly on B. (In Topology a function with these properties would be called a *strict Urysohn function* for the pair A, B.)

(v) If A, B are closed and disjoint, with $d(A, B) = 0$, a strict Urysohn function for the pair (A, B) cannot be uniformly continuous.

(vi) Solve the 'interpolation problem': given disjoint closed sets $A_1, \ldots A_N$ in R^n and constants $c_i \in R, i = 1, \ldots, N$, find $f : R^n \to R$ continuous and bounded, satisfying $f(x) = c_i$ for $x \in A_i$, $i = 1, \ldots, N$.

7. *Composition.* We know that if $f : A \to R^n$, $g : B \to R^p$ are continuous (with $A \subset R^m, B \subset R^n$ and $f(A) \subset B$, the composition $g \circ f : A \to R^p$ is continuous.

(i) If f, g are uniformly continuous on A resp. B, then $g \circ f$ is uniformly continuous on A.

(ii) If f, g are Lipschitz on A, B, with Lipschitz constants L, M (resp.), then $g \circ f$ is Lipshitz on A (with what Lipschitz constant?)

(ii) If f, g are Hölder continuous on A resp. B (with exponents α resp. β), then $g \circ f$ is Hölder continuous on A (with what exponent?)

8. *More on uniform continuity and almost periodic functions.* (See Anthony W. Knapp, *Basic Real Analysis*, p.131).

Denote by $UC_b(R)$ the space of bounded uniformly continuous functions $f : R \to R$, with the supremum norm.

(i) Show this defines a complete normed vector space, and contains all continuous periodic functions.

(ii) A function $f : R \to R$ is *(Bochner) almost-periodic* if the set of translates $\mathcal{T}_f = \{f_t; t \in R\}$, where $f_t(x) = f(x + t)$, is precompact (for the uniform norm.) Show that any continuous periodic function is 'almost-periodic'.

(iii) Prove: A function $f \in UC_b(R)$ is almost periodic if, and only if, the family $\mathcal{T}_f$ of translates is *totally bounded*: for any $\epsilon > 0$ we may find a finite subset $\{f_1, \ldots, f_N\} \subset \mathcal{T}_f$ so that any $f \in \mathcal{T}_f$ is ϵ-close to one of the f_i.

(In any metric space, a set is sequentially precompact iff it is totally bounded; and compact iff it is totally bounded and closed. Totally bounded implies bounded, but not conversely.)

(iv) Prove that the set of almost-periodic functions is an algebra, closed under uniform limits. (But note the set of *periodic* functions is not even a vector space.)

10/10/17, 23:50.