FUNCTIONS OF BOUNDED VARIATION

Remark. Throughout these notes, 'increasing' means 'nondecreasing':

$$x < y \Rightarrow f(x) \leq f(y).$$

1. Functions of bounded variation.

Definition. Given $f : [a, b] \to R$ and a partition

$$P = \{a = x_0 < x_1 < x_2 < \ldots < x_n < b = x_{n+1}\}$$

of $[a, b]$, the variation of f over P is;

$$V_P(f) = \sum_{j=0}^{n} |f(x_j) - f(x_{j+1})|.$$

f is of *bounded variation* if the numbers $V_P(f)$ form a bounded set, as P ranges over the set of all partitions of $[a, b]$. We denote the supremum of the $V_P(f)$ over all partitions P by $V_{ab}(f)$, the variation of f from a to b.

Let $BV[a, b]$ denote the real vector space of functions of bounded variation $f : [a, b] \to R$.

1) monotone functions, Lipschitz functions are in BV, but Hölder continuous functions are not always in BV (see Exercise 2). The indefinite Riemann integral of a (bounded) integrable function is in BV (why?)

2) Let $f(x) = x \cos \frac{\pi}{2x} < 0 < x \leq 1, f(0) = 0$. Consider the partition P of $[0, 1]$ given by $0 < \frac{1}{2n} < \frac{1}{2n-1} \ldots < \frac{1}{3} < \frac{1}{2} < 1$. The corresponding values of f are:

$$0, \frac{(-1)^n}{2n}, 0, \frac{(-1)^{n-1}}{n-1}, 0, \ldots, 0, -\frac{1}{6}, 0, \frac{1}{4}, 0, -\frac{1}{2}, 0,$$

and thus:

$$V_P(f) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

showing that f is not in $BV[0, 1]$.

3) $f \in BV[a, b] \Rightarrow f$ bounded.

4) The sum, difference and product of functions in BV is in BV; also $\frac{f}{g}$, if $f, g \in BV$ and g is bounded below by a positive constant.

5) $V_{ab}(f) = V_{ac}(f) + V_{cb}(f)$ if $a < c < b$.

Proof. If P_1, P_2 are partitions of $[a, c], [c, b]$ (resp.) and $P = P_1 \cup \{c\} \cup P_2$, we have $V_{ab}(P) = V_{ac}(P_1) + V_{bc}(P_2)$. This shows $V_{ac}(f) + V_{bc}(f) \leq V_{ab}(f)$.

Now let P be any partition of $[a,b]$. If $c \in P$, we have equality as above. If $c \notin P$, let P_1, P_2 be the partitions of $[a,c]$ (resp. $[c,b]$) consisting of points in P smaller than c (resp. greater than c). Then, since:

$$|f(x_{j+1}) - f(x_j)| \leq |f(c) - f(x_j)| + |f(x_{j+1}) - f(c)|$$

(where x_j is the last point of P_1 and x_{j+1} is the first point of P_2), we have:

$$V_P(f) \leq V_{P_1}(f) + V_{P_2}(f).$$

This proves the other inequality.

Exercise 1. Show that $V_{ab}(f+g) \leq V_{ab}(f) + V_{ab}(g)$ (for $f, g \in BV[a,b]$), and give an example where the inequality is strict.

Exercise 2. Let $C \subset [0,1]$ be Cantor's middle-thirds set, a closed, perfect subset of $[0,1]$. Let $\alpha \in (0,1)$. Then the function:

$$f(x) = d(x,C)^\alpha$$

is Hölder continuous with exponent α, but is not in $BV[0,1]$ if α is sufficiently small.

Outline. (i) In any complete metric space, the distance $d(x,C)$ from a point x to a compact set C is Lipschitz with constant 1. (*Hint:* Given $x, y \in X$, let $y_0 \in C$ achieve $d(y,C)$ and use $|x - y_0| \leq |x - y| + |y - y_0|$.)

(ii) Show that $|x^\alpha - y^\alpha| \leq |x - y|^\alpha$ for $x \geq 0, y \geq 0$, if $0 < \alpha < 1$.

Use (i) and (ii) to conclude f is Hölder continuous with exponent α. (Note that the distance is either zero, or achieved at some point $x_0 \in C$.)

(iii) At the k^{th} step of the construction of C, 2^{k-1} open intervals, each with length 3^{-k}, are deleted from the middle of a closed interval remaining in the previous step. Denote these intervals by $I_{kj}, 1 \leq j \leq 2^{k-1}, k \geq 1$. Show that the variation of f in I_{kj} equals:

$$V_{jk}(f) = 2(\frac{3^{-k}}{2})^\alpha.$$

Use this to compute the variation of f in $[0,1]$, and conclude it is unbounded if α is small enough.

(Reference: http://math.stackexchange.com/questions/1566955/are-there-functions-that-are-holder-continuous-but-whose-variation-is-unbounded.)

Proposition 1. For $f \in BV[a,b]$ let $v_f(x) = V_{ax}(f), x \in [a,b]$, be the *indefinite variation* of f ($v_f(a) = 0$), a monotone increasing function. Then

$v_f(x)$ and $v_f(x) - f(x)$ are both increasing in $[a, b]$. Thus any BV function may be written as the difference of two increasing functions.

Proof. Let $d_f(x) = v_f(x) - f(x)$, and consider two points $x < y$ in $[a, b]$. Then $d_f(y) - d_f(x) = V_{xy}(f) - (f(y) - f(x)) \geq 0$.

Corollary. Any $f \in BV[a, b]$ has at most countably many points of discontinuity, all of jump type; in addition, f is differentiable a.e in $[a, b]$.

In particular, any locally Lipschitz function (of one real variable) is differentiable a.e. (*Rademacher's theorem in one dimension.*)

2. Monotone and saltus functions.

Let $f : [a, b] \to R$ be monotone increasing. The *saltus* (jump) at $x \in (a, b)$ is $\Delta f(x) = f(x+) - f(x-)$. Also $\Delta f(a) = f(a+) - f(a), \Delta f(b) = f(b) - f(b-)$. (These are all nonnegative numbers.)We have, for any partition $P = \{x_j\}$ of $[a, b]$:

$$\Delta f(a) + \sum_{j=1}^{n} \Delta f(x_j) + \Delta f(b) \leq f(b) - f(a),$$

since we may find points $y_{j-1} < x_j < y_j$, $j = 1, \dots n$, in the open interval (a, b) so that:

$$\Delta f(x_j) \leq f(y_j) - f(y_{j-1}), \quad \Delta f(a) \leq f(y_0) - f(a), \quad \Delta f(b) \leq f(b) - f(y_n),$$

and then:

$$\Delta f(a) + \sum_{j=1}^{n} \Delta f(x_j) + \Delta f(b) \leq \sum_{j=1}^{n} f(y_j) - f(y_{j-1}) + f(y_0) - f(a) + f(b) - f(y_n) = f(b) - f(a).$$

Let f be monotone increasing on $[a, b]$. We define the *saltus function* $s_f : [a, b] \to R$ by

$$s_f(x) = f(a+) - f(a) + \sum_{\{a < x_k < x\}} \Delta f(x_k) + f(x) - f(x-), 0 < x \leq b; \quad s_f(a) = 0,$$

where the sum is over the countably many discontinuities of f that are less than x. (Note we already showed this sum is finite.) Clearly $s_f(x)$ is increasing and nonnegative. It is also easy to see that the image $s_f([a, b])$ is a countable set, and that s_f is *constant* (and thus differentiable, with zero derivative) on any open interval in which f is continuous.

Proposition 2. If f is increasing, the difference $c_f(x) = f(x) - s_f(x)$ is increasing and continuous in $[a, b]$.

Proof. Let $x < y$ be points in $[a, b)$. A short calculation shows that:

$$s_f(y)-s_f(x) = f(x+)-f(x)+\sum_{x<x_k<y} \Delta f(x_k)+f(y)-f(y-) \leq f(y)-f(x),$$

(*do it!*) This shows c_f is increasing. Taking limits as $y \to x+$, we get:

$$s_f(x+) - s_f(x) \leq f(x+) - f(x), \text{ or } c_f(x+) \geq c_f(x).$$

On the other hand, the same inequality shows that if $y > x$ we have:

$$s_f(y) - s_f(x) \geq f(x+) - f(x),$$

and letting $y \to x+$ we have: $f(x+) - f(x) \leq s_f(x+) - s_f(x)$, or $c_f(x+) \leq c_f(x)$. This shows c_f is right-continuous.

Exercise 3. Following similar steps, show $c_f(x-) = c_f(x)$, for $x \in (a, b]$.

Example. let $C = \{x_k\}_{k\geq 1}$ be any countable set in (a, b) (for example, C could be the set of rationals in(a, b).) Consider the function $f : [a, b] \to R$:

$$f(x) = \sum_{\{k; a<x_k<x\}} \frac{1}{2^k} \text{ for } 0 < x \leq b; \quad f(a) = 0.$$

Letting $I_k(x) = 1$ if $x_k < x$ and $x \in (a, b]$; $I_k(x) = 0$ if $x_k \geq x$; $I_k(a) = 0$ we have:

$$f(x) = \sum_{k=1}^{\infty} \frac{I_k(x)}{2^k}.$$

By the Weierstrass criterion, the series defining f is absolutely and uniformly convergent in $[a, b]$. f is clearly monotone increasing in $[a, b]$.

Claim. The set of discontinuity of f is exactly C, and that $\Delta f(x_k) = \frac{1}{2^k}$.

Outline. (i) If $x_0 \notin C$, f is continuous at x_0.

To show this, observe the series is uniformly convergent in $[a, b]$, and each function $\frac{I_k(x)}{2^k}$ is continuous on $[a, b] \setminus C$. Note that by uniform convergence given $\epsilon > 0$ we may find N so that:

$$|f(x) - f_N(x)| < \epsilon, x \in [a, b], \text{ where } f_N(x) = \sum_{k=1}^{N} \frac{I_k(x)}{2^k}.$$

Combine this inequality with continuity of f_N at x_0 to conclude.

(ii) To show f is discontinuous on C, note that if $x_{k_0} \in C$, we may write:

$$f(x) - \sum_{\{k; k \neq k_0\}} \frac{I_k(x)}{2^k} = \frac{I_{k_0}(x)}{2^{k_0}}.$$

By contradiction, assume f were continuous at x_{k_0}. Explain why this would imply continuity of the left-hand side at x_{k_0}, while the right-hand side is discontinuous at x_{k_0}.

(iii) To establish the claim about the jump at x_k, Let $(j_1, \ldots, j_N)$ be the permutation of $(1, \ldots, N)$ so that $a < x_{j_1} < \ldots < x_{j_N} < b$. Then show that if $x_{j_k} < x < x_{j_{k+1}}$:

$$f_N(x) = \frac{1}{2^{j_{k+1}}} + \frac{1}{2^{j_{k+2}}} + \ldots + \frac{1}{2^{j_N}}.$$

Show this implies $\Delta f_N(x_{j_k}) = \frac{1}{2^{j_k}}$, and explain how this leads to $\Delta f(x_{j_k}) = \frac{1}{2^{j_k}}$.

Remark: Note that the same argument implies that given any countable set $C = \{x_n; n \geq 1\}$, and any convergent series of positive numbers, $\sum_{n=1}^{\infty} a_n < \infty$ with $a_n > 0$, defining:

$$f(x) = \sum_{\{n \geq 1; x_n < x\}} a_n$$

we obtain a monotone increasing function with jumps $\Delta f(x_n) = a_n$ at points of C, continuous elsewhere, and with total variation $V(f) = \sum_{n \geq 1} a_n$ (on any sufficiently large bounded interval.) One can show this function is left-continuous at the x_n.

If $\sum a_n$ is absolutely convergent (but not of positive terms) we get a function of bounded variation, with given jumps at given points.

Now consider $f \in BV[a, b]$. f is the difference of two monotone increasing functions:

$$f(x) = v_f(x) - d_f(x).$$

So we define the saltus function of $f \in BV$ by:

$$s_f(x) = s_{v_f}(x) - s_{d_f}(x).$$

Proposition 3. If $f \in BV[a, b]$, f is the sum of its saltus function and a continuous function of bounded variation.

Proof. **Exercise 4.** (Use the decomposition 'saltus + continuous' of the increasing functions v_f and d_f).

Proposition 4. Let $f \in BV[a,b]$. If f is continuous at $x_0 \in (a,b)$, then v_f is continuous at x_0.

Proof. We show that if f is right-continuous at $x_0 \in [a,b)$, then so is v_f; (the proof for 'left-continuous' is analogous.)

Given $\epsilon > 0$, choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[x_0, b]$ so that $V_P(f) > V_{ab}(f) - \epsilon$. By right-continuity of f at x_0, we may assume $|f(x_0) - f(x_1)| < \epsilon$, for adding points to P would make $V_P(f)$ even larger. Letting P' be the partition starting at x_1, we have:

$$V_{x_0b}(f) < V_P(f) + \epsilon < V_{P'}(f) + 2\epsilon \leq V_{x_1b}(f) + 2\epsilon,$$

or:

$$V_{x_0x_1}(f) = v_f(x_1) - v_f(x_0) < 2\epsilon.$$

Since ϵ is arbitrary, this implies $v_f(x_1) \leq v_f(x_0)$. Since x_1 is an arbitrary point in (x_0, b), we conclude $v_f(x_0+) \leq v_f(x_0)$. But $v_f(x)$ is increasing, so in fact $v_f(x_0+) = v_f(x_0)$.

Corollary. Any continuous function of bounded variation is the difference of two continuous increasing functions.

3. Precompactness of sets in BV.

Theorem. (Helly's theorem.) Let $\mathcal{F}$ be an infinite family of functions in $BV[a,b]$, uniformly bounded ($|f(x)| \leq K$ in $[a,b]$, for all $f \in \mathcal{F}$, with uniformly bounded variation ($V_{ab}(f) \leq K$, for all $f \in \mathcal{F}$).

Then there is a sequence (f_n) in $\mathcal{F}$ converging pointwise in $[a,b]$ to a function of bounded variation.

Since any function in BV is the difference of two monotone functions, it is natural to reduce the proof to the increasing case. We have the lemma:

Lemma: Let $\mathcal{F}$ be an infinite family of increasing functions in $[a,b]$, uniformly bounded by K. Then there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{F}$, converging pointwise to an increasing function ϕ.

Proof. First, we have the *fact*: given any infinite, uniformly bounded family of functions $\mathcal{F}$ on $[a,b]$, and a countable subset $E \subset [a,b]$, one may find a sequence $(f_n)_{n\geq 1}$ in $\mathcal{F}$, converging pointwise in E (to a function $f : E \to R$.) This can be proved by a standard diagonal argument.

Applying this to the set E of rational points in $[a,b]$, we find a sequence (f_n) (and a corresponding infinite set $\mathcal{F}_0 \subset \mathcal{F}$) converging at every point of E (include a in E if needed.) Define ψ in E as the limit:

$$\psi(x) = \lim f_n(x), \quad x \in E.$$

It is easy to see ψ is increasing in E. On $[a,b] \setminus E$, define ψ by:

$$\psi(x) = \sup\{\psi(y); y \in E, y < x\}.$$

ψ is increasing on $[a,b]$, and hence continuous except on a countable set $Q \subset [a,b]$.

Claim: $\lim f_n(x_0) = \psi(x_0)$ at any x_0 where ψ is continuous.

Proof of claim: Given $\epsilon > 0$, using the continuity of ψ at x_0 we find x_l, x_r points of E so that $x_l < x_0 < x_r$, $\psi(x_r) - \psi(x_l) < \epsilon$. Then find a natural number N so that if $n \geq N$:

$$|f_n(x_r) - \psi(x_r)| < \epsilon, \quad |f_n(x_l) - \psi(x_l)| < \epsilon.$$

Then if $n \geq N$:

$$\psi(x_0) - 2\epsilon < \psi(x_l) - \epsilon < f_n(x_l) \leq f_n(x_0) \leq f_n(x_r) < \psi(x_r) + \epsilon < \psi(x_0) + 2\epsilon.$$

This proves the claim.

Thus $\lim f_n(x) = \psi(x)$ can only fail on the countable set Q where ψ is discontinuous. Now use the fact mentioned at the start of the proof, applied to the family $\mathcal{F}_0$ and the countable set Q. We obtain a subsequence (f_n) of $\mathcal{F}_0$ which converges at every point of $[a,b]$ to a function $\phi(x)$, clearly increasing. This concludes the proof of the lemma.

Important remark: The limit function may be discontinuous! (Can you think of an example?)

Proof of theorem. For each function $f \in \mathcal{F}$ we have the decomposition $f(x) = v_f(x) - d_f(x)$, both monotone. The family $\mathcal{F}_1 = \{v_f; f \in \mathcal{F}\}$ is uniformly bounded, so we may find a convergent sequence $(v_n) \subset \mathcal{F}_1$, $v_n = v_{f_n}$, $\lim v_n(x) = \nu(x)$, pointwise on $[a,b]$. (Note ν is increasing and bounded.)

Now apply the lemma to the sequence $(d_n) = (d_{f_n})$ of monotone functions. We find a subsequence d_{n_j} converging pointwise on $[a,b]$ to an increasing function $\delta(x)$. Then the sequence $f_{n_j}(x) = v_{n_j}(x) - d_{n_j}(x)$ converges pointwise on $[a,b]$ to the function of bounded variation $\nu(x) - \delta(x)$.

This concludes the proof of Helly's precompactness theorem.

Remark. We may define, for $f \in BV[a,b]$:

$$||f||_{BV} = \sup_{[a,b]} ||f|| + V_{ab}(f).$$

Exercise 5. Show that this defines a Banach norm in the vector space $BV[a,b]$ (i.e., BV with this norm is a complete metric space.)

Of course, Helly's theorem does not conclude convergence in this norm, but only pointwise.

4. Lebesgue outer measure in one dimension.

Given a subset $E \subset R$, we define its *Lebesgue outer measure* by:

$$m^*(E) = \inf\{\sum_{I \in \mathcal{F}} |I|; \mathcal{F} \text{ a countable family of open intervals, } E \subset \bigcup \mathcal{F}\}.$$

Properties.

(1) countably subadditive:

$$m^*(\bigcup_{n \geq q} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

Proof. Given $\epsilon > 0$, find intervals $(I_{n,m})$ so that:

$$E_n \subset \bigcup_{m \geq 1} I_{n,m}, \quad \sum_{m \geq 1} |I_{n,m}| \leq m^*(E_n) + \frac{\epsilon}{2^n}.$$

Let $\phi : N \to N \times N$ be a bijection. Then:

$$m^*(\bigcup_{n \geq 1} E_n) \leq \sum_{k=1}^{\infty} |I_{\phi(k)}| = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |I_{n,m}| \leq \sum_{n=1}^{\infty} (m^*(E_n) + \frac{\epsilon}{2^n}) = \sum_{n \geq 1} m^*(E_n) + \epsilon.$$

(2) It follows that $m^*(C) = 0$ if C is countable (since $m^*(\{x\}) = 0$ for all $x \in R$, as is easy to show.)

(3) If $I \subset R$ is an interval, $m^*(I) = |I|$.

Proof. It is enough to take $I = [a,b]$, and $m^*(I) \leq b - a$ is clear. Let $I \subset \cup \mathcal{F}$, where $\sum_{J \in \mathcal{F}} |J| < m^*(I) + \epsilon$. We may find $\mathcal{F}_1 \subset \mathcal{F}$ finite so that $I \subset \cup \mathcal{F}_1$.

Then inductively find intervals $(a_j, b_j) \in \mathcal{F}_1, j = 1, \ldots, p$, so that:

$$a_1 < a, \quad a_p < b < b_p, \quad a_n < b_{n-1} < b_n \text{ for } 2 \leq n \leq p.$$

Then:

$$|I| = b - a = b_1 - a + (\sum_{n=2}^{p-1}(b_n - b_{n-1})) + b - b_{p-1} \leq \sum_{n=1}^{p}(b_n - a_n) \leq m^*(I) + \epsilon.$$

(4) Translation invariance: $m^*(E + t) = m^*(E)$ for any $t \in R$ (since $|I + t| = |I|$ for each interval I.)

Remark. (The problem with Lebesgue outer measure.) $m^*(E)$ is defined for all subsets $E \subset R$. Unfortunately, m^*, although countably subadditive, is not even 'finitely additive': there exist *disjoint* subsets $A, B \subset R$ so that:

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

The solution to this is to define a class of "measurable subsets" of R, restricted to which m^* is, indeed, countably additive: the measure of a countable disjoint union (of measurable sets) is the sum of the measures of the sets.

5. Cantor sets and Lebesgue's singular function.

We can generalize the usual construction of Cantor's middle-thirds set (a compact, perfect subset of $[0, 1]$) by varying the lengths of the open intervals removed at the nth. step of the construction. Thus, consider a sequence $(a_n)_{n\geq 0}$ of positive real numbers, satisfying:

$$a_0 = 1, \quad 0 < a_{n+1} < \frac{1}{2}a_n, n \geq 0.$$

(In the classical construction, $a_n = 3^{-n}$). At the zero-th step, there is only one interval, $J_{0,1} = [0, 1]$. At the nth. step ($n \geq 1$) we remove the middle open interval (with length $d_n = a_{n-1} - 2a_n$) of each of the 2^{n-1} closed intervals remaining at the $(n-1)^{th}$. step (each with length a_{n-1}). This yields 2^n closed intervals,each with length a_n.

Denote by $\{J_{n,k}; 1 \leq k \leq 2^n\}$ the closed intervals remaining at the nth. step, and let $P_n = \cup_{k=1}^{2^n} J_{n,k}$ be their union. Let $I_{n,k}; 1 \leq k \leq 2^{n-1}$ be the open intervals removed at the n^{th} step. Thus:

$$P_{n-1} \setminus P_n = \bigcup_{k=1}^{2^{n-1}} I_{n,k}.$$

For instance,

$$J_{1,1} = [0, a_1], \quad J_{1,2} = [1 - a_1, 1], \quad I_{1,1} = (a_1, 1 - a_1)$$

$$J_{2,1} = [0, a_2], \quad J_{2,2} = [a_1 - a_2, a_2], \quad I_{2,1} = (a_2, a_1 - a_2),$$
$$J_{2,3} = [1 - a_1, 1 - a_1 + a_2], \quad J_{2,4} = [1 - a_2, 1], \quad I_{2,2} = (1 - a_1 + a_2, 1 - a_2).$$

The decreasing intersection of compact sets:

$$P = \bigcap_{n=1}^{\infty} P_n$$

is a compact, non-empty subset of $[0, 1]$ (and in fact uncountable, without isolated points): the Cantor set defined by the sequence (a_n).

Proposition 4. $m^*(P) = \lim 2^n a_n$ (a monotone decreasing sequence).

Proof. For each $n \geq 1$, $m^*(P) \leq m^*(P_n) \leq \sum_{k=1}^{2^n} m^*(J_{n,k}) = 2^n a_n$. This shows $m^*(P) \leq \lim 2^n a_n$. On the other hand:

$$m^*([0, 1] \setminus P) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} m^*(I_{n,k}) = \sum_{n=1}^{\infty} 2^{n-1} d_n$$

$$= \sum_{n=1}^{\infty} 2^{n-1}(a_{n-1} - 2a_n) = \sum_{n=1}^{\infty} (2^{n-1} a_{n-1} - 2^n a_n) = 1 - \lim 2^n a_n.$$

Since $1 = m^*([0, 1]) \leq m^*(P) + m^*([0, 1] \setminus P)$, or $m^*(P) \geq 1 - m^*([0, 1] \setminus P)$, this shows $m^*(P) \geq \lim 2^n a_n$.

Remark. Note this implies that for any $0 \leq \lambda < 1$, we may choose (a_n) so that $m^*(P) = \lambda$ (for instance, let $a_n = \frac{\lambda}{2^n} + \frac{1-\lambda}{3^n}$.) For the classical middle-thirds set, the outer Lebesgue measure is zero.

Lebesgue's singular monotone function. We construct $\psi : R \to [0, 1]$ continuous, monotone increasing (nondecreasing), surjective, with zero derivative a.e.

Recall the deleted open intervals $\{I_{n,k}; 1 \leq k \leq 2^{n-1}\}$ in the construction of the classical middle-thirds Cantor set ($a_n = 3^{-n}, m^*(P) = 0$). We define P to be constant in each $I_{n,k}$, as follows:

$$\psi = \frac{1}{2} \text{ on } I_{1,1}; \quad \psi = \frac{1}{4} \text{ on } I_{2,1}; \quad \psi = \frac{3}{4} \text{ on } I_{2,2};$$

$$\psi = \frac{2k - 1}{2^n} \text{ on } I_{n,k}, k = 1, \ldots 2^{n-1}.$$

This defines ψ in $[0, 1] \setminus P$, a union of open intervals contained in $[0, 1]$. Then extend ψ to $[0, 1]$ via:

$$\psi(0) = 0, \quad \psi(x) = \sup\{\psi(t); t \in [0, 1] \setminus P, t < x\} \text{ for } x \in (0, 1].$$

Finally, extend to R by setting $\psi(x) = 0$ for $x \leq 0$; $\psi(x) = 1$ for $x \geq 1$. Clearly ψ is increasing (nondecreasing).

Claim. ψ is continuous.

Proof. If $x \in (0,1)$ and $\psi(x-) < \psi(x+)$, then the image $\psi([0,1])$ misses the interval $(\psi(x-), \psi(x+))$. But that's impossible since the image of ψ includes all the dyadic rationals $\frac{2k-1}{2^n}$, a dense subset of $[0,1]$.

To show right-continuity at 0, let x_k be any sequence in $(0,1)$ converging to 0. Then given $n \geq 1$ we may find K_n so that $x_k < \frac{1}{2^n}$ for $k \geq K_n$, so $x_k < y_n$ for some $y_n \in I_{n,1}$, and by monotonicity:

$$\psi(x_k) \leq \psi(y_n) = \frac{1}{2^n} \text{ for } k \geq K_n.$$

Given $\epsilon > 0$, we choose n so that $\frac{1}{2^n} < \epsilon$, and then $0 \leq \psi(x_k) \leq \epsilon$ for $k \geq K(\epsilon)$. Thus $\lim_k \psi(x_k) = 0 = \psi(0)$.

Exercise 6. Show that $\psi(1) = 1$, and that ψ is left-continuous at 1.

Remark: ψ is onto $[0,1]$ by a similar argument.

Being monotone, ψ is differentiable a.e. And $\psi' = 0$ in $U = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}$, an open set whose complement P has zero outer Lebesgue measure in $[0,1]$. Thus $\psi' = 0$ a.e. in R.

In addition, since $\psi(U)$ is countable (dyadic rationals) it is a set of measure zero. But this means $\psi(P)$ cannot have measure zero, since ψ is onto: $[0,1] = \psi(U) \cup \psi(P)$. So ψmaps a set of measure zero to one which is not of measure zero.

(In fact letting m^* be outer Lebesgue measure on the line we have $m^*(\psi(U)) = 0$ and hence:

$$1 = m^*([0,1]) = m^*(\psi([0,1]) \leq m^*(\psi(U)) + m^*(\psi(P)) = m^*(\psi(P)) \leq 1,$$

so $m^*(\psi(P)) = 1$.)

With a little more work, one can get a *strictly increasing* continuous function, with zero derivative a.e. (see the next section.)

6. Fubini's theorem on series of monotone functions.

Let $(f_n)_{n \geq 1}$ be a sequence of increasing functions on $[a,b]$. Assume the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges pointwise in $[a, b]$. Then f is differentiable a.e. and $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ a.e.

Application 1: There exists a strictly increasing continuous function $f : R \to R$, such that $f'(x) = 0$ a.e.

Let $\psi(x) : R \to [0, 1]$ be Lebesgue's continuous increasing function with zero derivative a.e. (ψ is constant in each open interval in the complement of the Cantor set in $[0, 1]$, in $(1, \infty)$ and in $(-\infty, 0)$.)

Let $([a_n, b_n])_{n=1}^{\infty}$ be an enumeration of the set of all non-degenerate closed intervals with rational endpoints.

Define $f_n : R \to [0, 2^{-n}]$ by $f_n(x) = 2^{-n}\psi(\frac{x-a_n}{b_n-a_n})$; f_n is continuous and increasing.

Then set $f(x) = \sum_{n=1}^{\infty} f_n(x)$. By the Weierstrass M-test, this series converges uniformly on R to a continuous increasing function.

If $x < y$ are two real numbers, we may find an $n \geq 1$ so that $x < a_n < b_n < y$. Then:

$$f(y) - f(x) \geq f_n(y) - f_n(x) \geq f_n(b_n) - f_n(a_n) = \frac{1}{2^n}(\psi(1) - \psi(0)) > 0.$$

Thus f is strictly increasing.

By Fubini's theorem, $f'(x) = \sum f_n'(x) = 0$ a.e. (Note f and each f_n are differentiable a.e.)

Application 2. More generally, let

$$f(x) = \sum_{\{n \geq 1; x_n < x\}} a_n = \sum_{n=1}^{\infty} a_n I_n(x)$$

be the BV function associated to a countable subset $C = \{x_n; n \geq 1\}$ of (a, b) and an absolutely convergent series $\sum_{n \geq 1} a_n$ (see the remark after Exercise 4). Then $f'(x) = 0$ a.e., since each function $a_n I_n(x)$ has zero derivative a.e.

In fact if (x_n) is dense in (a, b) this function is strictly increasing: if $x < y$, let n_0 be an index so that $x < x_{n_0} < y$. Then it is easy to check that $f(x) + a_{n_0} \leq f(y)$.

Proof of Fubini's theorem.

Denote by s_k, r_k the partial sum and remainder ($k \geq 1$):

$$s_k(x) = \sum_{n=1}^{k} f_n(x), \quad r_k(x) = \sum_{n=k+1}^{\infty} f_n(x).$$

Both are increasing functions on $[a, b]$(as is f). Let $E \subset [a, b]$ be the set where f and all the f_n are differentiable. The complement of E in $[a, b]$ has measure zero.

Since r_k is increasing, we have $0 \leq r'_k(x) = f'(x) - s'_k(x), x \in E$.

(i) First we show that if a series of functions increasing on a set E, and differentiable on E, converges pointwise in E, then its series of derivatives converges (pointwise in E). Since $f'_n(x) \geq 0$ in E for each n, we have:

$$s'_k(x) \leq s'_{k+1}(x) \leq f'(x), x \in E.$$

Thus:

$$\sum_{n=1}^{\infty} f'_n(x) = \lim s'_k(x) \leq f'(x).$$

(ii) For the reverse inequality, choose $k_1 < k_2 < \ldots$ so that:

$$\sum_{j=1}^{\infty} (f(b) - s_{k_j}(b)) < \infty.$$

(Say $s_{k_j}(b) > f(b) - 2^{-j}$, using $s_k(b) \uparrow f(b)$.) Since $0 \leq f(x) - s_{k_j}(x) = r_{k_j}(x) \leq r_{k_j}(b)$, we see that $\sum_{j=1}^{\infty}(f(x) - s_{k_j}(x))$ converges uniformly on $[a, b]$.

This is a series of increasing functions in $[a, b]$, and *by the reasoning in part (i) of this proof* the corresponding series of derivatives converges!

$$\sum_{j=1}^{\infty} (f'(x) - s'_{k_j}(x)) < \infty.$$

In particular, $s'_{k_j}(x) \to f'(x)$ as$J \to \infty$, for $x \in E$. Since the series of partial sum derivatives $s'_k(x)$ is increasing k, it must converge to $f(x)$, for $x \in E$.

7. Rectifiable curves.

A parametrized curve $\alpha : [a, b] \to R^n$ (not necessarily continuous) is *rectifiable* if:

$$L[\alpha] := \sup_{P \in \mathcal{P}} L_P[\alpha] < \infty, \text{ where } L_P[\alpha] := \sum_{j=0}^{n} ||\alpha(t_{j+1}) - \alpha(t_j)||$$

and the *sup* is taken over the set $\mathcal{P}$ of all finite partitions $P = \{t_0 = a < t_1 < \ldots < t_n < t_{n+1} = b\}$ of $[a, b]$ and we use the euclidean norm.

Geometrically $L_P[\alpha]$ is the length of the polygonal line defined by the points on the curve $p_j = \alpha(t_j)$. Thus $L[\alpha]$ corresponds to the length, if α is rectifiable.

Note this corresponds exactly to the definition of bounded variation: rectifiable parametrized curves are functions of bounded variation taking values in R^n; so we adopt the notation $BV([a,b];R^n)$.

In terms of components in the standard basis of R^n, $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$, where each $\alpha_k : [a,b] \to R$. In fact, considering:

$$|\alpha_k(t_{j+1}) - \alpha_k(t_j)| \leq ||\alpha(t_{j+1}) - \alpha(t_j)|| \leq \sum_{i=1}^{n} |\alpha_i(t_{j+1}) - \alpha_i(t_j)|, \quad k = 1, \ldots, n,$$

we see that a parametrized curve α is rectifiable if and only if each component function is in $BV[a,b]$.

In particular, if α is rectifiable the tangent vector $\alpha'(t)$ exists a.e. in $[a,b]$. The *arc length function* of a rectifiable curve is the indefinite variation:

$$s(t) = L[\alpha_{|[a,t]}], \quad t \in [a,b]; \quad s(b) = L[\alpha].$$

The arc length is monotone increasing (and continuous when α is), in particular differentiable a.e.

Exercise 7. Show that $s'(x) = ||\alpha'(x)||$ for a.e. $x \in [a,b]$. (In fact s is differentiable at a given $x \in (a,b)$ if and only if α is differentiable at x.)

Hint: Let $x \in (a,b)$ be a point where α is differentiable. Given $\epsilon > 0$, choose $\delta > 0$ so that

$$\alpha(x+h) = \alpha(x) + h\alpha'(x) + hr(h), \text{ with } ||r(h)|| < \epsilon \text{ if } 0 < |h| < \delta.$$

Let $\{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = 1\}$ be an arbitrary partition of $[0,1]$. Show that:

$$\sum_{k=0}^{n} ||\alpha(x + t_{k+1}h) - \alpha(x + t_k h)|| \leq |h|(||\alpha'(x)|| + 2\epsilon)$$

and explain why this implies s is differentiable at x with $s'(x) \leq ||\alpha'(x)||$.

The opposite inequality is easier to prove.

Exercise 8. Show that $f \in BV[a,b]$ if and only if the graph $\gamma(t) = (t, f(t))$ of f is rectifiable, and:

$$V_{ab}(f) \leq L[\gamma] \leq b - a + V_{ab}(f).$$

Given its geometric origin, it is natural to expect the length of a rectifiable curve is independent of the choice of parametrization. A reparametrization is defined by a homeomorphism $h : [c, d] \to [a, b]$ (which we take to be increasing, so $h(c) = a, h(d) = b$). Such an h defines a bijection h_* between the sets of finite partitions of $[c, d]$ and of $[a, b]$:

$$h_* : \mathcal{P}_{[c,d]} \to \mathcal{P}_{[a,b]}, \quad h(t_i) = x_i, i = 1, \dots, n, \quad t_i \in [c, d], x_i \in [a, b].$$

Thus, with $\alpha : [a, b] \to R^n$, $\beta = \alpha \circ h : [c, d] \to R^n$, we have, for any $P \in \mathcal{P}_{[a,b]}$:

$$L_{h_*\mathcal{P}}(\beta) = \sum_{i=0}^{n} ||\beta(t_{i+1}) - \beta(t_i)|| = \sum_{i=0}^{n} ||\alpha(x_{i+1}) - \alpha(x_i)|| = L_P(\alpha).$$

From this it follows that $\beta = \alpha \circ h$ is rectifiable if and only if α is, with the same length: $L[\alpha] = L[\beta]$, and similarly for the arc length functions: $s_\beta = s_\alpha \circ h$.

Remark. h is monotone, hence $h'(t)$ exists in a subset $[c, d] \setminus N$, where $N \subset [c, d]$ has measure zero. We also know $\alpha'(x)$ exists in $[a, b] \setminus E$, where $E \subset [a, b]$ has measure zero; and that $\beta'(t)$ exists outside of a subset of $[c, d]$ with measure zero. It is natural to ask whether

$$\beta'(t) = \alpha'(h(t))h'(t),$$

where $t \in [c, d]$ is such that both terms on the right are defined, namely: $t \in ([c, d] \setminus N) \cap \{t \in [c, d]; h(t) \in [a, b] \setminus E\}$. For this set to have complement of measure zero, we need the preimage $h^{-1}(E)$ to have measure zero. This isn't true in general, but it does hold if h is a Lipschitz homeomorphism. (The precise condition needed is given in the next section).

Exercise 9. Show that if $h : [c, d] \to [a, b]$ is a (strictly increasing) bi-Lipschitz homeomorphism (meaning the inverse is also Lipschitz), and $E \subset [a, b]$ is a set of measure zero, then the preimage $h^{-1}(E)$ has measure zero.

Example. Let $\psi : [0, 1] \to [0, 1]$ be the continuous, increasing function with $\psi'(x) = 0$ a.e. constructed in Section 5. Note that the preimage of the set of dyadic rationals in $[0, 1]$ (which is countable, hence of measure zero) contains the complement in $[0, 1]$ of the Cantor set, which has (outer) measure one.

Exercise 10. Let $\mathcal{F} \subset BV([a, b]; R^n)$ be an infinite family of rectifiable curves in R^n. Suppose $\mathcal{F}$ is uniformly bounded ($\alpha([a, b]) \subset B_R \forall \alpha \in \mathcal{F}$, for some ball $B_R \subset R^n$) and of uniformly bounded length ($L[\alpha] \leq \Lambda$ for all $\alpha \in \mathcal{F}$, for some $\Lambda > 0$.)

Show there exists a sequence $(\alpha_n)_{n\geq 1}$ in $\mathcal{F}$ converging pointwise to a rectifiable curve $\alpha : [a,b] \to R^n$. (Note the limit curve α may be discontinuous, or constant.)

8. Absolutely continuous functions.

Definition. $f : [a,b] \to R$ is *absolutely continuous* if for all $\epsilon > 0$ we may find $\delta > 0$ so that if $\{(a_k, b_k)\}_{k=1}^n$ is a finite collection of *disjoint* open intervals contained in $[a,b]$, we have:

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

Properties.

1. Letting $n = 1$, we see that absolutely continuous functions are uniformly continuous.

2. AC functions define a vector subspace of $C[a,b]$, in fact an algebra ($fg \in AC$ if f and g are AC); f/g is AC if f, g are and g is bounded below by a positive constant. Lipschitz functions are clearly AC.

3. AC functions are of bounded variation. To see this, let $f \in AC[a,b]$ and let $\delta > 0$ be given by the definition for $\epsilon = 1$. Let $a = a_0 < a_1 < \ldots < a_n < b$ be any finite partition of $[a,b]$ into intervals of length less than δ (that is, $a_{k+1} - a_k < \delta$. Then from the definition of AC it follows that $V_{a_k a_{k+1}}(f) \leq 1$, so $V_{ab}(f) \leq n$.

4. Lebesgue's singular continuous function $\psi : [0,1] \to [0,1]$ is in $BV[0,1]$ (since it is increasing), but not in $AC[0,1]$. To see this, recall $P_n = \sqcup\{J_{nk}; 1 \leq k \leq 2^n\}$, the disjoint union of closed intervals (each of length 3^{-n}) in the n^{th} stage of the construction of Cantor's middle-thirds set $P \subset [0,1]$.

Using the continuity of ψ and its constant value $(2k-1)2^{-n}$ in the open interval I_{nk}, a short computation shows that if $J_{nk} = [a_k, b_k]$ we have $\psi(b_k) - \psi(a_k) = 2^{-n}$, for $k = 1, \ldots 2^n$.

Thus we have $\sum_{k=1}^{2^n}(b_k - a_k) = (\frac{2}{3})^n$ (which can be made arbitrarily small by taking n large enough), while $\sum_{k=1}^{2^n} |f(b_k) - f(a_k)| = 2^n 2^{-n} = 1$.

Extensions. (i) It is easy to see that f is AC if (and only if) $\forall\epsilon > 0 \exists\delta > 0$

so that if $\{(a_k, b_k\}_{k=1}^{\infty}$ is a collection of disjoint sub-intervals of $[a, b]$, we have:

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} M_k - m_k < \epsilon,$$

where

$$M_k = \sup_{x\in[a_k,b_k]} f(x) = f(\beta_k), \quad m_k = \inf_{x\in[a_k,b_k]} f(x) = f(\alpha_k), \quad \alpha_k, \beta_k \in [a_k, b_k].$$

(ii) It is also easy to see we may take a countable (infinite) collection of disjoint intervals $\{(a_k, b_k)\}_{k\geq 1}$ in the definition, without changing it.

Proposition 5. If $f \in AC[a, b]$ and $E \subset [a, b]$ is a set of measure zero ($m^*(E) = 0$), we have $m^*(f(E)) = 0$.

Proof. (Assume $E \subset (a, b)$ for simplicity). Given $\epsilon > 0$, choose $\delta > 0$ (based on extensions (i) and (ii) above) so that if $\{(\alpha_j, \beta_j)\}_{j\geq 1}$ is a countable collection of disjoint intervals in (a, b), we have:

$$\sum_{j=1}^{\infty}(\beta_j - \alpha_j) < \delta \Rightarrow \sum_{j=1}^{\infty} M_j - m_j < \epsilon,$$

with M_j, m_j the sup (resp. inf) of f on (a_j, b_j), as above.

Since $m^*(E) = 0$, we have $E \subset \cup_{n\geq 1} I_n$, where the I_n are open intervals contained in (a, b) and $\sum_{n\geq 1} |I_n| \leq \delta$.

Since $U = \cup_{n\geq 1} I_n$ is an open subset of the real line, it may be written as a countable disjoint union of open intervals:

$$U = \sqcup_{j=1}^{\infty}(\alpha_j, \beta_j), \text{ with } \sum_{j=1}(\beta_j - \alpha_j) = m^*(U) \leq \sum_{n=1}^{\infty} |I_n| < \delta,$$

where the first equality uses *countable additivity* of outer Lebesgue measure on open intervals, while the inequality following it uses only countable subadditivity of outer measure (which holds for all sets).

Noting that $f((\alpha_j, \beta_j)) = (m_j, M_j)$ (the sup and inf of f on (α_j, β_j)), we find a countable covering by open intervals:

$$f(E) \subset \bigcup_{j=1}^{\infty}(m_j, M_j), \text{ with } \sum_{j=1}^{\infty}(M_j - m_j) < \epsilon.$$

This shows $m^*(f(E)) = 0$.

Additional facts on absolute continuity. (For proofs, see I.Natanson, *Theory of Functions of a Real Variable*, ch. 9.)

1. Conversely, if a continuous function of bounded variation maps null sets (=sets of measure zero) to null sets, it is absolutely continuous. (*Banach-Zarecki theorem.*)

2. An absolutely continuous function f has derivatives a.e. (since it is BV). If $f'(x) = 0$ a.e., then f is constant.

3. A continuous function maps (Lebesgue) measurable sets to measurable sets if and only if it maps null sets to null sets.

4. (i) If $g : [a, b] \to R$ is integrable, its indefinite integral $f(x) = \int_a^x g(t)dt$ is absolutely continuous, and $f'(x) = g(x)$ a.e.

(ii) Conversely, if f is absolutely continuous, $f'(x)$ is integrable and we have:

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

Informally, we may think of AC as exactly the class of continuous functions on $[a, b]$ for which the Fundamental Theorem of Calculus (both directions) holds. (For the Lebesgue integral.)