DIFFERENTIAL FORMS and LINE INTEGRALS

Let $U \subset R^n$ be a connected open set. A *1-form* in U is a map $\omega : U \to \Lambda^1$, where Λ^1 denotes the space of linear functionals in R^n (i.e. linear maps $R^n \to R$). In standard coordinates:

$$\omega(x) = \sum_i a_i(x) dx_i, \quad dx_i[v] = v_i,$$

the i^{th} component of v. ω is said to be continuous (or C^k) in U if each component $a_i \in C(U)$ (resp. $a_i \in C^k(U)$).

Let $\gamma : [a, b] \to U$ be a rectifiable curve. We define the *line integral* of the 1-form ω along γ by:

$$\int_\gamma \omega = \int_0^1 \omega(\gamma(t))[\gamma'(t)]dt.$$

Proposition. (Invariance under increasing reparametrization). Let $f : [a, b] \to [c, d]$ be a C^1 diffeomorphism, $f'(t) > 0, f(a) = c, f(b) = d$. Then if $\alpha : [c, d] \to U$ is a rectifiable curve and $\gamma = \alpha \circ f : [a, b] \to U$, we have:

$$\int_\gamma \omega = \int_\alpha \omega.$$

Exact 1-forms. A continuous $1 - form$ is *exact* in the connected open set U if there exists $f \in C^1(U)$ so that $df = \omega$ in U. Such an f (which is unique up to a constant) is called a *potential* for ω.

The following is one version of the Fundamental Theorem of Calculus in several variables.

Theorem. (i) Let $f \in C^1(U)$. Then if $\omega = df \in C_U(\Lambda^1)$, for any rectifiable curve $\gamma : [a, b] \to U$:

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a)).$$

Thus line integrals of ω along a curve depend only on the endpoints of the curve.

(ii) Conversely, if a 1-form $\omega \in C_U(\Lambda^1)$ with the property that its line integrals along curves in U depend only on the endpoints of the curve is exact in U. A potential is obtained by choosing a point $x_0 \in U$ and defining:

$$f(x) = \int_\gamma \omega, \quad \gamma : [0, 1] \to U, \gamma(0) = x_0, \gamma(1) = x.$$

A condition clearly equivalent to that stated in (ii) is: $\int_\gamma \omega = 0$, for amy *closed* (rectifiable) curve γ in U.

A 1-form ω in U is *locally exact* if, for any $x_0 \in U$, there exists an open ball $B = B_r(x_0) \subset U$ and a potential $f \in C^1(B)$ for ω on this ball. This is equivalent to: (i) line integrals of ω along curves in B depend only on their endpoints; or (ii) line integrals of ω along closed curves in B are zero.

Definition. A 1-form $\omega = \sum a_i dx_i$ of class C^1 in U is *closed* if:

$$\partial_{x_j} a_i - \partial_{x_i} a_j = 0 \text{ in } U, \forall i, j.$$

From Schwartz's theorem for C^2 functions, it is easy to see that locally exact C^1 1-forms are closed. The converse is false:

$$\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

is closed in $R^2 \setminus \{0\}$, but not exact there, since its line integral along the unit circle equals 2π. On the other hand, ω is exact in the half-plane $H = \{(x,y) \in R^2; x > 0\}$: $f(x,y) = \arctan(y/x)$ is a potential for ω in H.

Definition. A connected open set $U \subset R^n$ is *starshaped* if there exists $x_0 \in U$ so that, for any $x \in U$, the closed line segment from x_0 to x is entirely contained in U. (For example, convex domains are starshaped with respect to any interior point.)

Poincaré Lemma. If $U \subset R^n$ is starshaped, any closed 1-form $\omega \in C^1_U(\Lambda^1)$ is exact in U.

For the proof, one defines $f(x)$ as the line integral of ω along the line segment from x_0 to x, using the fact that ω is closed (and differentiation under the integral sign) to show f is a potential for ω in U.

As a *corollary*, it follows that a 1-form of class C^1 is closed if, and only if, it is locally exact.

Problem 1. Is the locally uniform limit of locally exact forms locally exact? (That is, if $\omega_k \to \omega$ uniformly on compact subsets of U (ω_k, ω continuous 1-forms in U), and each ω_k is locally exact, is the same true for ω?

Question: Conversely: is any continuous, locally exact 1-form the limit (uniformly on compact sets) of *closed* 1-forms of class C^1?

The precise topological concept relating local to global exactness (of continuous 1-forms) is given by *homotopy*: two continuous curves $\gamma, \alpha : [0,1] \to U$ ($\gamma(0) = \alpha(0) = p, \gamma(1) = \alpha(1) = q$) are *homotopic in U with fixed endpoints* if there exists a continuous map $H : [0,1]^2 \to U$ satisfying:

$$H(0,t) = \gamma(t), H(1,t) = \alpha(t), H(s,0) = p, H(s,1) = q.$$

So the 1-parameter family of curves $\gamma_s(t) = H(s,t)$ 'deforms' $\gamma_0(t) = \gamma(t)$ to $\gamma_1(t) = \alpha(t)$, through continuous curves in U as s varies from 0 to 1.

For closed curves $\gamma(0 = \gamma(1) = p$ the analogous definition (continuous deformation through closed curves with basepoint p) is called *homotopy with basepoint.*

Proposition. (Homotopy invariance of line integrals.) If γ and α are fixed-endpoint-homotopic (rectifiable) curves in U and ω is a locally exact, continuous one-form in U, then:

$$\int_\gamma \omega = \int_\alpha \omega.$$

Definition. A connected open set $U \subset R^n$ is *simply-connected* if any two continuous curves $\gamma, \alpha : [0,1] \to U$ are homotopic with fixed endpoints. (Equivalently, any closed curve with base point p is basepoint-homotopic in U to the constant curve p.)

Corollary. In a simply-connected domain, any locally exact continuous 1-form is exact.

Conversely, given a connected open set $U \subset R^n$, if any locally exact 1-form is exact, it follows that U is simply connected.

Differential 2-forms. A bilinear form $\alpha : R^n \times R^n \to R$ is *alternating* if $f(w,v) = -f(v,w)$ for all v, w; equivalently, $f(v,v) = 0$ for all v. The set Λ^2 of alternating bilinear forms is a vector space of dimension $n(n-1)/2$ ($n \geq 2$), with standard basis $\{dx_i \wedge dx_j\}_{i<j}$:

$$(dx_i \wedge dx_j)[v,w] = v_i w_j - v_j w_i, \text{ where } v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n).$$

In particular, the space $\Lambda^2(R^2)$ is one-dimensional: any bilinear alternating form in R^2 is a multiple of $det[v,w] = \det[v|w]$, the determinant of the 2×2 matrix given by column vectors v, w.

A differential 2-form is a map $\alpha : U \to \Lambda^2$:

$$\alpha(x) = \sum_{i<j} b_{ij}(x) dx_i \wedge dx_j,$$

where the b_{ij} are continuous (or C^k) functions in U.

Wedge product. Given α, β in Ω^1_U (1-forms in U) we define their 'wedge product' as the 2-form:

$$(\alpha \wedge \beta)[v, w] = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

Clearly $\beta \wedge \alpha = -\alpha \wedge \beta$.

Exterior differential. Let $\omega = \sum_i a_i dx_i \in \Omega^1_U$ be of class C^1. We define:

$$d\omega = \sum_i da_i \wedge dx_i = \sum_{i,j} (\partial_{x_j} a_i) dx_j \wedge dx_i \in \Omega^2_U.$$

This is easily seen to equal:

$$d\omega = \sum_{i<j} (\partial_{x_j} a_i - \partial_{x_i} a_j) dx_j \wedge dx_i.$$

We adopt this expression as the definition of $d\omega$. Thus we see that

$$\omega \text{ is closed } \Leftrightarrow d\omega = 0 \text{ in } U.$$

Problem 2. (i) $d(f\omega) = df \wedge \omega + f d\omega$ if $\omega \in \Omega^1_U$ is C^1 and $f \in C^1(U)$.

(ii) A 2-form $\alpha \in \Omega^2_U$ is *exact* in U if $\alpha = d\omega$, for some C^1 1-form ω in U. Show that if α is closed and β is exact, $\alpha \wedge \beta$ is exact. (Here $\alpha, \beta \in \Omega^1_U$ are C^1).

Pullback. Let $f : D \to U$ be a C^1 map ($y = f(x)$, where $D \subset R^m, U \subset R^n$ are connected open sets. We define maps from 1 and 2-forms in U to 1 and 2-forms in D.

For 1-forms $\omega \in \Omega^1_U$, $f^*\omega \in \Omega^1_D$ is defined by:

$$f^*\omega(x)[v] = \omega(f(x))[df(x)v], \quad x \in D, v \in R^n.$$

In coordinates:

$$\omega(y) = \sum_i a_i(y) dy_i \Rightarrow f^*\omega(x) = \sum_{i=}^{n} a_i(f(x)) df_i$$

$$= \sum_{a=1}^{m} b_a(x) dx_a, \quad b_a(x) = \sum_i (\partial_{x_a} f_i)(x) a_i(f(x)), \quad x \in D, f(x) \in U.$$

For 2-forms $\alpha = \sum_{i<j} b_{ij}(y) dy_i \wedge dy_j$, define $f^*\alpha \in \Omega^2_D$ by:

$$f^*\alpha(x)[v,w] = \alpha(f(x))[df(x)[v], df(x)[w]], \quad x \in D, v, w \in R^n.$$

In coordinates:

$$f^*\alpha(x) = \sum_{i<j} b_{ij}(f(x)) df_i \wedge df_j$$

$$= \sum_{a<b} A_{ab} dx_a \wedge dx_b, \quad A_{ab} = \sum_{i<j} b_{ij}(f(x))(\partial_{x_a} f_i \partial_{x_b} f_j - \partial_{x_b} f_i \partial_{x_a} f_j).$$

Proposition. (Invariance of exterior derivative.) For $f : D \to U$ of class C^2 and $\omega \in \Omega^1_U$ a C^1 1-form as above, we have:

$$f^* d\omega = df^*\omega.$$

Proof. A calculation (done in class.)

Corollary. $f^*\omega$ is closed in D if ω is closed in U. The converse holds if f is a diffeomorphism.

Integration of 2-forms in R^2. Let $\alpha \in \Omega^2_U$, $U \subset R^2$ open. If $\phi : D \to U$ is a C^1 diffeomorphism, the pullback of α is given by:

$$\alpha = f(y) dy_1 \wedge dy_2 \Rightarrow \phi^*\alpha = (f \circ \phi) d\phi_1 \wedge d\phi_2 = (f \circ \phi) J_\phi(x) dx_1 \wedge dx_2,$$

assuming ϕ is 'orientation-preserving', in the sense that $\det d\phi(x) > 0$ for all $x \in D$.

This suggests the *definition*: for $A \subset U$ measurable:

$$\int_A \alpha = \int_A f(y) dy$$

(Integration with respect to Lebesgue measure, assuming f has an integral.) Then the change of variables formula implies the transformation formula:

$$\int_B \phi^*\alpha = \int_{\phi(B)} \alpha,$$

if $B \subset D$ is measurable. Recall also the transformation formula for the line integral of 1-forms ω along curves γ in D:

$$\int_\gamma \phi^*\omega = \int_{\phi \circ \gamma} \omega.$$

Suppose we know that, for a subset $A \subset\subset U \subset R^2$ with piecewise C^1 oriented boundary ∂A, we have, for any 1-form ω of class C^1 in U:

$$\int_A d\omega = \int_{\partial A} \omega.$$

Then if $\phi : D \to U$ is a diffeomorphism we have $Q = \phi^{-1}(A) \subset\subset D$, a domain with piecewise C^1 (oriented) boundary. Given a C^1 1-form η in D we have $\eta = \phi^*\omega$ for a 1-form ω in U, so:

$$\int_{\partial Q} \eta = \int_{\partial\phi(Q)} \omega = \int_{\phi(Q)} d\omega = \int_Q \phi^* d\omega = \int_Q d\eta.$$

So Q satisfies the same property as A (invariance of Stokes' theorem.)

Stokes' theorem for the rectangle. Let $Q = [0,1]^2 \subset R^2$, $c = \partial Q$ its oriented boundary. Let $\omega \in \Omega^1_U$ be a 1-form of class C^1 in a neighborhood U of Q. Then:

$$\int_c \omega = \int_Q d\omega.$$

Homotopy estimate. Let $c_0, c_1 : [0,1] \to U \subset R^2$ be C^1 curves with common endpoints, $c_0(0) = c_1(0), c_0(1) = c_1(1)$. Suppose c_0, c_1 are pointwise close enough that the line segment from $c_0(t)$ to $c_1(t)$ is contained in U. Then the curves are linearly homotopic, via:

$$H : [0,1] \to U, \quad H(s,t) = (1-s)c_0(t) + sc_1(t).$$

Then if $\omega \in \Omega^1_U$ is a C^1 1-form, let:

$$H^*\omega = Pds + Qdt, \quad d(H^*\omega) = (Q_s - P_t) ds \wedge dt.$$

We have, with c the oriented boundary of $[0,1]^2$:

$$\int_c H^*\omega = \int_{c_1} \omega - \int_{c_0} \omega.$$

$$dH^*\omega = H^* d\omega = (d\omega(H(s,t))[H_s, H_t]) ds \wedge dt.$$

By Stokes' theorem for the rectangle:

$$\int_c H^*\omega = \int_{[0,1]^2} (Q_s - P_t) ds dt,$$

where, for $K = H([0,1]^2) \subset U$ (compact):

$$|Q_s - P_t| = |d\omega(H(s,t))[H_s, H_t]| \leq (\sup_K |d\omega|) \sup_{t\in[0,1]} |c_0(t) - c_1(t)|(s|c_1'|(t) + (1-s)|c_1'(t)).$$

Thus we obtain the *homotopy estimate*:

$$|\int_{c_0} \omega - \int_{c_1} \omega| \leq (\sup_K |d\omega|) \sup_{t\in[0,1]} |c_0(t) - c_1(t)|(L[c_0] + L[c_1]).$$

As a consequence, we have:

Proposition. Let $c_k : [0,1] \to U$ be a sequence of rectifiable curves of bounded length (with common endpoints), converging uniformly in $[0,1]$ to a rectifiable curve $c_0 : [0,1] \to U$. Then for any C^1 1-form ω, we have:

$$\int_{c_k} \omega \to \int_{c_0} \omega.$$

Remark: It is interesting that this is true even though the lengths of the c_k may fail to converge to the length of c_0.

Problem 3. [Giaquinta-Modica] Let ω be a closed C^1 1-form in $R^n \setminus \{0\}$. Prove that ω is exact in $R^n \setminus \{0\}$ if $\lim_{x\to 0} |x|\omega(x) = 0$.

Problem 4. [Giaquinta-Modica] (i) Let ω be a closed 1-form in $R^2 \setminus \{0\}$ satisfying:

$$\int_\gamma \omega = 0,$$

where $\gamma(t) = (\cos t, \sin t), t \in [0, 2\pi]$. Show that ω is exact in $R^2 \setminus \{0\}$.

(ii) Show that any closed 1-form ω in $R^2 \setminus \{0\}$ decomposes as:

$$\omega = \lambda\omega_0 + \alpha,$$

where $\lambda \in R$, α is exact in $R^2 \setminus \{0\}$ and ω_0 is the (closed) 'angle 1-form' in $R^2 \setminus \{0\}$:

$$\omega_0 = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$