JORDAN CONTENT

Definition. Let $A \subset R^n$ be a bounded set. Given a rectangle (cartesian product of compact intervals) $R \subset R^n$ containing A, denote by $\mathcal{P}$ the set of finite partitions of R by sub-rectangles ('intervals') defined by some grid. Consider the numbers:

$$\underline{J}(P, A) = \sum \{m(I_k); I_k \text{ an interval of } P \text{ contained in } int(A)\}$$

$$\bar{J}(P, A) = \sum \{m(I_k); I_k \text{ an interval of } P \text{ intersecting } cl(A)\}.$$

Here $int(A)$ and $cl(A) = A \cup \partial A$ denote the interior and the closure of A (resp.) Clearly $0 \leq \underline{J}(P, A) \leq \bar{J}(P, A)$, and we set $\underline{J}(P, A) = 0$ for any P if A has empty interior.

We define the inner and outer Jordan content of A by taking sup (resp inf):

$$\underline{c}(A) = \sup\{\underline{J}(P, A); P \in \mathcal{P}\}; \quad \bar{c}(A) = \inf\{\bar{J}(P, A); P \in \mathcal{P}\}.$$

A is *Jordan-measurable* if these two numbers coincide: $\underline{c}(A) = \bar{c}(A) := c(A)$, the *Jordan content* of A.

It is easy to see that the inner and outer Jordan content are independent of the choice of reference rectangle R containing A.

The following is clear geometrically, and easy to prove. For any partition $P \in \mathcal{P}$ of R, and any $A \subset R$:

$$\bar{J}(P, A) - \underline{J}(P, A) = \bar{J}(P, \partial A).$$

This implies the following.

Proposition 1. Let $A \subset R^n$ be a bounded set. Then $\bar{c}(A) - \underline{c}(A) = \bar{c}(\partial A)$. In particular, A is Jordan-measurable if and only if $\bar{c}(\partial A) = 0$.

Proof. Let R be a rectangle containing A; consider an arbitrary partition P of R. Then:

$$\bar{J}(P, \partial A) = \bar{J}(P, A) - \underline{J}(P, A) \geq \bar{c}(A) - \underline{c}(A);$$

since this holds for all P, we have $\bar{c}(\partial A) \geq \bar{c}(A) - \underline{c}(A)$.

For the opposite inequality, given $\epsilon > 0$ consider partitions P_1, P_2 of R so that:

$$\bar{c}(A) \geq \bar{J}(P_1, A) - \epsilon, \quad \underline{c}(A) \leq \underline{J}(P_2, A) + \epsilon.$$

Let P be a partition of R refining both P_1 and P_2. Refinement decreases $\bar{J}$ and increases $\underline{J}$, so:

$$\bar{c}(\partial A) \leq \bar{J}(P, \partial A) = \bar{J}(P, A) - \underline{J}(P, A) \leq \bar{J}(P_1, A) - \underline{J}(P_2, A) \leq \bar{c}(A) - \underline{c}(A) + 2\epsilon,$$

and now let $\epsilon \to 0$.

Remarks.

(1) The sets of rational numbers and of irrational numbers in an interval $[a, b] \subset R$ both have inner content zero, and outer content $b - a$ (*Check this.*) This example also shows that neither the inner nor the outer Jordan content is additive (over disjoint sets.)

(2) However, Jordan content is finitely additive: if A, B are disjoint Jordan-measurable sets, then $A \cup B$ is Jordan-measurable and $c(A \cup B) = c(A) + c(B)$

Problem. (i) Let $A, B \subset R^n$ be bounded sets. We have:
$\bar{c}(A \cup B) + \bar{c}(A \cap B) \leq \bar{c}(A) + \bar{c}(B)$.
(Analogously $\underline{c}(A \cup B) + \underline{c}(A \cap B) \geq \underline{c}(A) + \underline{c}(B)$.)

(ii) If $A, B \subset R$ are bounded and Jordan measurable, then $A \cup B$ and $A \cap B$ are Jordan measurable and

$$c(A \cup B) + c(A \cap B) = c(A) + c(B).$$

(It is also true that If A and B are Jordan measurable and $B \subset A$, then $A \setminus B$ is Jordan measurable and $c(A \setminus B) = c(A) - c(B)$.)

(3) We have the following connection with the Riemann integral: the outer (resp. inner) Jordan content of a bounded set A equals the upper (resp. lower) Riemann integral of its characteristic function.

Question 1. Is a countable union of (disjoint) Jordan measurable sets Jordan measurable? Is Jordan content countably additive?

Question 2. Is every bounded open subset of R^n Jordan-measurable?

Question 3. Let $C \subset [0, 1]$ be the standard Cantor set, $A = [0, 1] \setminus C$. Is A Jordan-measurable? What if C has positive outer Lebesgue measure?

LEBESGUE MEASURE. (Following [Fleming, ch.5])

Outer measure. For a bounded set $A \subset R^n$, define:

$$m^*(A) = \inf\{\sum_{k \geq 1} vol(R_k); A \subset \bigcup_{k \geq 1} R_k\},$$

where the infimum is taken over all countable coverings of A by *open* rectangles R_k.

Problem 1. For every bounded set $A \subset R^n$ we have $m^*(A) \leq \bar{c}(A)$.

Hint: Let P be any partition of a rectangle R containing A. Consider the rectangles R_k of P that intersect $A \cup \partial A$. Then, by definition:

$$\bar{J}(P, A) = \sum_{k \geq 1} vol(R_k).$$

Let $\epsilon > 0$ be arbitrary. Show that by enlarging the R_k slightly we obtain a countable covering $\{R'_k\}_{k\geq 1}$ of A by open rectangles, satisfying:

$$\sum_{k \geq 1} vol(R'_k) = \bar{J}(P, A) + \epsilon.$$

Problem 2. Let $A \subset R^n$ be a bounded set, $K \subset A$ a compact set. Then $\bar{c}(K) \leq m^*(A)$.

Hint: Let R be a rectangle containing A; let $\epsilon > 0$ be given. Take a countable open covering $\{R_k\}_{k\geq 1}$ of A so that $\sum_{k\geq 1} vol(R_k) < m^*(A) + \epsilon$. Show that a finite subcover of K determines a partition P of R for which $\bar{J}(P, K) \leq m^*(A) + \epsilon$. Conclude.

Corollary. If $K \subset R^n$ is compact, then $\bar{c}(K) = m^*(K)$.

We next define *Lebesgue measure* of bounded sets in R^n. First we define measure for bounded open sets, and for compact sets. This is based on approximating from the inside (resp. from the outside) by *figures*, where a *figure* P is a finite union of closed rectangles $R_k, k = 1, \dots N$. We may assume the rectangles comprising P have disjoint interior, and then the *volume* $vol(P)$ of P is defined in the obvious geometric way.

Definitions.

(i) Let $U \subset R^n$ be a bounded open set. We define its measure by:

$$m(U) = \sup\{vol(P); P \subset U \text{ a figure}\}.$$

Problem 0.1. Show that, for a bounded open set U: $m(U) = \underline{c}(U)$, the lower Jordan content.

If $K \subset R^n$ is compact, we define:

$$m(K) = \inf\{vol(P), P \text{ a figure }, K \subset int(P)\}.$$

Problem 0.2. Show that, for a compact set K: $m(K) = \bar{c}(K)$, the upper Jordan content.

Consistency check: It is easy to verify that $m(P) = vol(P)$ and $m(intP) = vol(P)$, if P is a figure.

(iii) Let $A \subset R^n$ be bounded. We define the *outer measure* of A approximating from the outside by open sets:

$$\overline{m}(A) = \inf\{m(U); A \subset U, U \text{ open }\}.$$

(iv) Let $A \subset R^n$ be bounded. We define the *inner measure* of A approximating from the inside by compact sets:

$$\underline{m}(A) = \sup\{m(K); K \subset A, K \text{ compact }\}.$$

Remark: Since, for K compact and U open, we always have $K \subset U \Rightarrow m(K) \leq m(U)$ (*Justify!,*) we have, for all bounded A: $\underline{m}(A) \leq \bar{m}(A)$.

(v) We say a bounded set $A \subset R^n$ is *measurable* if $\underline{m}(A) = \bar{m}(A)$, and then set $m(A)$ to be their common value.

Problem 0.3 We now have potentially two definitions of outer measure (of a bounded set). Show that these definitions are equivalent: $\bar{m}(A) = m^*(A)$, for any bounded set $A \subset R^n$.

Consistency check.

Problem 3. Bounded open sets U are Lebesgue measurable, and $m(U) = \underline{c}(U)$, the lower Jordan content.

Problem 4. Compact sets K are Lebesgue measurable, and $m(K) = \bar{c}(K)$, the upper Jordan content.

Question 4. Is every (bounded) Jordan-measurable set Lebesgue-measurable?
Hint: Show that we have, for all bounded A:

$$\underline{c}(A) \leq \underline{m}(A) \leq \bar{m}(A) \leq \bar{c}(A).$$

Cauchy criterion. $A \subset R^n$ (bounded) is measurable if, and only if, for all $\epsilon > 0$ we may find $K \subset A \subset U$ (K compact, U open) so that $m(U \setminus K) < \epsilon$.

This uses the fact that $K \subset U$ implies $m(U) = m(K) + m(U \setminus K)$.

First properties. (See [Fleming, 5.2] for proofs.)

1. (Finite additivity.) $A, B \subset R^n$ bounded, measurable, disjoint $\Rightarrow$ $A \sqcup B$ measurable, and:

$$m(A \sqcup B) = m(A) + m(B).$$

2. $A, B \subset R^n$ measurable $\Rightarrow A \setminus B, A \cap B, A \cup B$ measurable and:

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

For unbounded sets $A \subset R^n$, we say A is measurable if $A \cup B_R$ is measurable, for each $R > 0$ (open ball of radius B), and set $m(A) = \lim_{R \to \infty} m(A \cap B_R)$. We allow the possibility $m(A) = +\infty$.

3. A measurable $\Rightarrow A^c$ measurable (complement).

4. (Countable additivity) If $(A_n)_{n \geq 1}$ is a countable family of measurable sets, then $\bigcup_{n \geq 1} A_n$ and $\bigcap_{n \geq 1} A_n$ are measurable, and we have:

$$m(\bigcup_{n \geq 1} A_n) \leq \sum_{n \geq 1} m(A_n),$$

with equality if the sets A_n are disjoint.

5. (Limits.) (i) If $(A_n)_{n \geq 1}$ are measurable and $A_n \subset A_{n+1}$, then $m(\bigcup_{n \geq 1} A_n) = \lim m(A_n)$.

(ii) If $(A_n)_{n \geq 1}$ are measurable and $A_n \supset A_{n+1}$ *and* $m(A_1) < \infty$, then $m(\bigcap_{n \geq 1} m(A_n)) = \lim m(A_n)$.

RIEMANN INTEGRAL

Let $R \subset R^n$ be a rectangle, $f : R \to \mathbb{R}$ bounded. Given a finite partition $P = \{I_k\}_{k=1}^N$ of R, define upper and lower Riemann sums by:

$$U(P, f) = \sum_{k=1}^N M_k m(I_k), \quad L(P, f) = \sum_{k=1}^N m_k m(I_k), \quad M_k = \sup_{I_k} f, m_k = \inf_{I_k} f.$$

Definition.f is *Riemann integrable* if for all $\epsilon > 0$ we may find a partition P of R so that:

$$U(P, f) - L(P, f) < \epsilon.$$

Analogously to the Lebesgue theory, we define upper and lower Riemann integrals by taking inf and sup (resp.) of upper and lower Riemann sums over the set $\mathcal{P}$ of finite partitions of R:

$$\overline{\int} f dV = \inf\{U(P, f); P \in \mathcal{P}\}; \quad \underline{\int} f dV = \sup\{L(P, f); P \in \mathcal{P}\}.$$

It is easy to show that f is Riemann integrable if and only if upper and lower Riemann integrals coincide; the common value is the Riemann integral of f:

$$f \text{ integrable } \Leftrightarrow \overline{\int} f dV = \underline{\int} f dV := \int f dV.$$

(We use dm for the Lebesgue integral, dV for the Riemann integral.)

Riemann integrability criteria. For $f : R \to \mathbb{R}$ ($R \subset R^n$ a rectangle) denote by $\omega_f(x)$ the oscillation of f at x. Let $D_f \subset R$ be the discontinuity set, and denote by D_ϵ the closed set:

$$D_\epsilon = \{x \in R; \omega_f(x) \geq \epsilon\}, \text{ so } D_f = \{x \in R; \omega_f(x) > 0\} = \bigcup_{n \geq 1} D_{1/n}.$$

Theorem. *Jordan content criterion.* $f : R \to \mathbb{R}$ (bounded) is Riemann integrable if, and only if, $\bar{c}(D_\epsilon) = 0$, $\forall \epsilon > 0$.

Proof. (i) Assume first $\bar{c}(D_\delta) \neq 0$ for some $\delta > 0$. Let P be a partition of R. Then:

$$U(P, f) - L(P, f) = \sum_{k=1}^{N} [M_k - m_k] m(I_k) \geq \sum_{\{k; I_k \cap D_\delta \neq \emptyset\}} [M_k - m_k] m(I_k) > \delta \bar{J}(P, D_\delta),$$

using the facts that (i) $M_k - m_k \geq \delta$ if $I_k \cap D_\delta \neq \emptyset$; (ii) $\bar{J}(P, D_\delta)$ equals the sum of $m(I_k)$ over all k so that $I_k \cap D_\delta \neq \emptyset$, since D_δ is closed.

Thus we have a positive lower bound $\delta \bar{c}(D_\delta)$ (independent of P) for the difference $U(P, f) - L(P, f)$; so f can't be Riemann integrable over R.

(ii) Conversely, assume $\bar{c}(D_\epsilon) > 0$ for all $\epsilon > 0$. Given $\epsilon > 0$, let P_0 be a partition of R so that $\bar{J}(P_0, D_\epsilon) < \epsilon$. In each rectangle I of P not intersecting D_ϵ (or contained in the complement D_ϵ^c) we have $\omega_f(x) < \epsilon$ for $x \in I$. It follows by a Lebesgue number argument that there exists a $\delta > 0$ so that if $I \subset D_\epsilon^c$ and $diam(I) < \delta$ then the oscillation of f in I is smaller than ϵ.

Thus we refine P_0 so that all intervals of the refined partition $P_\epsilon = \{I_k\}_{k=1}^n$ have diameter smaller than δ. Let $A_1 = \{k; I_k \cap D_\epsilon \neq \emptyset\}, A_2 = \{k; I_k \subset D_\epsilon^c\}$. Then:

$$U(P_\epsilon, f) - L(P_\epsilon, f) = (\sum_{k \in A_1} + \sum_{k \in A_2})[M_k - m_k] m(I_k).$$

The sum over A_1 is bounded above by $(M-m)\bar{J}(P_\epsilon, D_\epsilon) \leq (M-m)\bar{J}(P_0, D_\epsilon)$, where M, m are the sup (resp. inf) of f in R. For the terms in the A_2 sum, as seen in the previous paragraph we have $M_k - m_k < \epsilon$. We conclude the difference $U - L$ is bounded above by $(M-m)\epsilon + \epsilon m(R)$, and hence can be made as small as desired. This shows f is Riemann-integrable.

Vitali's theorem. $f : R \to \mathbb{R}$ (bounded) is Riemann integrable if, and only if, $m^*(D_f) = 0$.

Proof. (i) Assume f is Riemann-integrable. Then $m^*(D_\epsilon) \leq \bar{c}(D_\epsilon) = 0$ for all $\epsilon > 0$ (Problem 1 above), hence by subadditivity of m^*:

$$m^*(D_f) \leq \sum_{n \geq 1} m^*(D_{1/n}) = 0.$$

(ii) Conversely, if $m^*(D_f) = 0$, since $D_\epsilon \subset D_f$ and D_ϵ is compact (for all $\epsilon > 0$), it follows that (Problem 2) $\bar{c}(D_\epsilon) \leq m^*(D_f) = 0$ for all $\epsilon > 0$. Thus f is Riemann integrable.

Now if $A \subset R^n$ is a bounded set and $f : A \to \mathbb{R}$ is bounded, we let R be any rectangle containing R in its interior and consider the extension $f_A : R \to \mathbb{R}$ of f to R defined by setting $f_A(x) = 0$ for $x \in R \setminus A$. We say f is *integrable over A* if f_A is integrable in R. Note that the discontinuity set of f_A is contained in the union $D_f \cup \partial A$. We conclude:

Corollary. Let $A \subset R^n$ be bounded. If $m^*(\partial A) = 0$ and $f : A \to \mathbb{R}$ is bounded, with $m^*(D_f) = 0$, then f is integrable over A.

Remark: Note that since ∂A is compact, the condition $m^*(\partial A) = 0$ is equivalent to saying A is Jordan measurable.