LEBESGUE DIFFERENTIATION THEOREM

For $f \in L^1_{loc}(R^n)$, consider its average over the ball $B_r(x)$:

$$f_r(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)dy.$$

It is easy to show $\lim_{r \to 0_+} f_r(x) = f(x)$ if f is continuous.

Theorem. (*Lebesgue differentiation theorem.*) If $f \in L^1_{loc}$, $\lim_{r \to 0_+} f_r(x) = f(x)$ for a.e x.

Proof. It is enough to assume $f \in L^1$. Note that f_r is obtained from f as the convolution:

$$f_r(x) = \frac{1}{r^n} \int_{R^n} k(\frac{x-y}{r}) f(y)dy, \quad k(z) = \frac{1}{\omega_n} \chi_{B_1}(z),$$

the characteristic function of the unit ball (divided by the volume ω_n of the unit ball.) So we know $f_r \to f$ in L^1, and in particular a subsequence f_{r_i} converges to f pointwise a.e. So the result follows if we can show the 'local oscillation':

$$\Omega_f(x) := \limsup_{r \to 0_+} f_r(x) - \liminf_{r \to 0_+} f_r(x)$$

equals zero a.e. We also know that given $\epsilon > 0$ we may write:

$$f = g + \phi, \quad \phi \in C_c(R^n), \quad ||g||_1 < \epsilon.$$

And we have, for all x:

$$\Omega_f(x) \leq \Omega_g(x) + \Omega_\phi(x) = \Omega_g(x),$$

since $\Omega_\phi \equiv 0$ if ϕ is continuous.

Suppose we can bound (for any $g \in L^1$) the distribution function of Ω_g by the L^1 norm of g:

$$m\{x; \Omega_f(x) > \alpha\} \leq m\{x; \Omega_g(x) > \alpha\} < c(\alpha)||g||_1, \quad \forall \alpha > 0.$$

Then it follows that the distribution function of Ω_f is arbitrarily small:

$$\lambda_{\Omega_f}(\alpha) := m\{x; \Omega_f(x) > \alpha\} < c(\alpha)\epsilon \quad \forall \epsilon > 0,$$

so $\Omega_f(x) = 0$ a.e., as we wished to show.

The estimate for the distribution function of Ω_f follows from an important estimate for the *Hardy-Littlewood maximal function:*

$$M_f(x) := \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|(y)dy, \quad f \in L^1_{loc}(R^n).$$

Theorem. (i) If $f \in L^p$ with $1 < p \leq \infty$, then $||M_f||_p < c_p||f||_p$, where c_p depends only on p and n, and blows up as $p \to 1_+$. (In particular, $M_f(x)$ is finite a.e.)

(ii) For $f \in L^1$, we have the 'weak (1,1) estimate':

$$\lambda_{M_f}(\alpha) := m\{x; M_f(x) > \alpha\} \leq \frac{A}{\alpha}||f||_1, \quad \forall \alpha > 0.$$

(Where A depends only on n).

The desired estimate for the distribution function of the oscillation Ω_f then follows from the easy observation $\Omega_f(x) \leq 2M_f(x)$, leading to:

$$\lambda_{\Omega_f}(\alpha) \leq \lambda_{M_f}(\alpha/2) \leq \frac{2A}{\alpha}||f||_1.$$

The main step in the proof of the 'weak (1,1) estimate' for the maximal function is a covering theorem of Vitali type:

Theorem. (*Vitali covering.*) Let $E \subset R^n$ be a non-empty measurable subset, $E \subset \bigcup_{B \in \mathcal{F}}$ a covering of E by a collection $\mathcal{F}$ of closed balls with bounded radius. Then there exists a countable subcollection $\{B_k\}_{k \geq 1}$ of *disjoint* balls so that $\sum_k m(B_k) \geq Cm(E)$.

Here C depends only on n (we may take $C = 5^{-n}$).

Proof of the weak (1,1) estimate for M_f.

Given $\alpha > 0$, let $E_\alpha = \{x; M_f(x) > \alpha\}$. If $x \in E_\alpha$ we may find a ball $B(x)$ with center x so that:

$$m(B(x)) \leq \frac{1}{\alpha} \int_{B(x)} |f|(y)dy \leq \frac{1}{\alpha}||f||_1.$$

This defines a covering of E_α by closed balls of bounded radius, and the covering theorem gives a countable subcollection $\{B(x_k)\}_{k \geq 1}$ so that:

$$C\lambda_{M_f}(\alpha) = Cm(E_\alpha) \leq \sum_k m(B(x_k)) \leq \frac{1}{\alpha} \sum_k \int_{B(x_k)} |f(y)|dy \leq \frac{1}{\alpha}||f||_1,$$

where the last inequality follows from the fact the collection $\{B(x_k)\}_k$ is disjoint. This is the desired $(1, 1)$ estimate.

Proof of the L^p estimate for M_f, $p > 1$.
We reduce it to the L^1 case by considering, for given $\alpha > 0$, the function:

$$f_1(x) = f(x) \text{ if } |f(x)| > \alpha/2; \quad f_1(x) = 0 \text{ otherwise.}$$

Exercise 1. $f_1 \in L^1(R^n)$ if $f \in L^p(R^n)$, $p \geq 1$.

Clearly $|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$, so $M_f(x) \leq M_{f_1}(x) + \frac{\alpha}{2}$ and, for their distribution functions:

$$\lambda_{M_f}(\alpha) \leq \lambda_{f_1}(\alpha/2) \leq \frac{2A}{\alpha}||f_1||_1 = \frac{2A}{\alpha} \int_{\{|f(x)|>\alpha/2\}} |f(x)|dx.$$

Next we need the expression of the L^p norm in terms of distribution functions. For $p = 1$, recall the 'Cavalieri formula':

$$\int |f(x)|dx = \int_0^\infty \lambda_f(\alpha)d\alpha, \quad \lambda_f(\alpha) := m\{x; |f(x)| > \alpha\}.$$

By a change of variable argument (for the Riemann integral in one variable) we have, for $1 \leq p < \infty$:

$$\int |f(x)|^p dx = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)d\alpha.$$

Proof. **Exercise 2.**

Combining this with the earlier estimate for λ_{M_f} and using Fubini's theorem we obtain:

$$\int |M_f(x)|^p dx \leq 2Ap \int_0^\infty \alpha^{p-2} \int_{\{|f(x)|>\alpha/2\}} |f(x)|dxd\alpha$$

$$= 2Ap \int_{R^n} (\int_0^{2|f(x)|} \alpha^{p-2} d\alpha)|f(x)|dx = \frac{2^p Ap}{p-1} \int_{R^n} |f(x)|^p dx.$$

This is the desired L^p estimate for M_f.