MEASURABLE FUNCTIONS

Notation. $F$ is the $\sigma$-algebra of Lebesgue-measurable subsets of $X = \mathbb{R}^n$. Given $E \in F$, $f : E \to \mathbb{R}$ (the extended real line) and $\alpha \in \mathbb{R}$ we adopt the notation:

$$E\{f > \alpha\} = \{x \in E; f(x) > \alpha\} = f^{-1}((\alpha, +\infty)) \in F.$$ 

In this handout we adopt the notations $m(E), m^*(E), m_*(E)$ for Lebesgue measure, outer and inner Lebegue measure (resp.)

Definition. $f$ is measurable if for any $\alpha \in \mathbb{R}$, $E\{f > \alpha\}$ is in $F$.

It is easy to see that this is equivalent to requiring measurability of one of the following types of sets, for any $\alpha$:

$$E\{f \geq \alpha\}, \quad E\{f < \alpha\}, \quad E\{f \leq \alpha\}.$$

It is also equivalent to requiring:

(i) For each open set $A \subset \mathbb{R}$, $f^{-1}(A) \in F$,

or to requiring

(ii) For each closed set $F \subset \mathbb{R}$, $f^{-1}(F) \in F$.

Recall that an algebra of subsets of $X$ is a family $\mathcal{F}$ of subsets with the properties (i) $X$ and $\emptyset$ are in $\mathcal{F}$; (ii) If $A \in \mathcal{F}$, the complement $A^c = X \setminus A$ is also in $\mathcal{F}$; (iii) If $A, B \in \mathcal{F}$, then $A \cup B, A \cap B$ and $A \setminus B$ are also in $\mathcal{F}$.

A family of subsets of $X$ is a $\sigma$-algebra if it is an algebra and is closed under countable union:

$$A_n \in \mathcal{F} \text{ for } n = 1, 2, \ldots \Rightarrow \bigcup_{n \geq 1} A_n \in \mathcal{F}.$$ 

(It follows that $\mathcal{F}$ is also closed under countable intersection.)

Given any family $\mathcal{G}$ of subsets of $X$, consider the intersection of all $\sigma$-algebras containing $\mathcal{G}$. This is again a $\sigma$-algebra, the $\sigma$-algebra generated by $\mathcal{G}$. The Borel subsets of $\mathbb{R}^n$ is the $\sigma$-algebra generated by the family of open subsets of $\mathbb{R}^n$. (Note this depends only on the topology, not on any measure.) Since open sets are Lebesgue-measurable, it follows that the Borel $\sigma$-algebra is contained in the $\sigma$-algebra of Lebesgue-measurable sets. In fact we have:

Fact: $E \subset \mathbb{R}^n$ is (Lebesgue) measurable if and only if there exists a Borel set $B \supset E$ with $m^*(B \setminus E) = 0$, if and only if there exists a Borel set $C \subset E$ with $m^*(E \setminus C) = 0$. 

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Problem 1. (Preimages behave nicely.) (i) Let \( f : E \to \bar{R} \) be a function. Show that the family of subsets \( \{ A \subset \mathbb{R}; f^{-1}(A) \in \mathcal{F} \} \) is a \( \sigma \)-algebra of subsets of \( R \).

(ii) Show that if \( f : E \to \bar{R} \) is measurable, then for every Borel set \( B \subset R \) we have \( f^{-1}(B) \in \mathcal{F} \).

Surprisingly, we have:

Example. There are measurable functions \( f : \mathbb{R} \to \mathbb{R} \) and (Lebesgue) measurable sets \( E \subset \mathbb{R} \) such that \( f^{-1}(E) \) is not measurable.

Let \( \phi : [0, 1] \to [0, 1] \) be Lebesgue’s singular function. The function \( g(x) = x + \phi(x) : [0, 1] \to [0, 2] \) is invertible, and maps the standard Cantor set (which has measure zero) onto a set of positive measure. And it is a fact that any set of positive measure contains a non-measurable set.

In fact, the following is true: a function \( f : \mathbb{R} \to \mathbb{R} \) maps measurable sets to measurable sets if, and only if, \( f \) maps sets of measure zero to sets of measure zero. ([Natanson, p.248, Theorem 2]).

Problem 2. (i) Let \( f, g : E \to \bar{R} \) be measurable. Then \( \max\{f, g\} \) and \( \min\{f, g\} \) are measurable. In particular, \( f_+ = \max\{f, 0\}, f_- = \max\{-f, 0\} \) and \( |f| = f_+ + f_- \) are measurable.

(ii) Let \( f : E \to \bar{R} \) be measurable and \( \phi : R \to R \) be continuous. Then the composition \( \phi \circ f \) is measurable. (In particular \( |f|^p \) (for any \( p \in \mathbb{R} \)) and \( e^f \) are measurable.)

Pointwise and a.e. limits. Let \( f_n, f : E \to \bar{R} \). Suppose \( f_n(x) \to f(x) \) pointwise in \( E \). Then \( f \) is measurable if each \( f_n \) is. To see this, let

\[
A_m^k = E\{f_k \geq \alpha + \frac{1}{m}\}, \quad B_m^n = \bigcap_{k=n}^{\infty} A_m^k.
\]

Then it is easy to see that:

\[
E\{f > \alpha\} = \bigcup_{m \geq 1, n \geq 1} B_m^n,
\]

and the set on the right is clearly measurable.

The same holds if we only know \( f_n \to f \) a.e. in \( E \): there is a null set \( N \subset E \) such that \( f_n \to f \) in \( E \setminus N \). Thus \( f \) is measurable in \( E \setminus N \), and also in \( N \) (since \( m(N) = 0 \), so \( f \) is measurable in \( E \)).

Proposition 1. (Lebesgue). Let \( E \subset X \) be measurable, with \( m(E) < \infty \). Suppose \( f_n \to f \) a.e. in \( E \), where \( f_n, f \) are measurable in \( E \) and finite a.e.
Then we have, for each $\sigma > 0$;

$$\lim_n m(E_n(\sigma)) = 0,$$

where $E_n(\sigma) = \{x \in E; |f_n(x) - f(x)| \geq \sigma\}$.

**Remark:** $m(E) < \infty$ is needed here: consider $f_n : R \to R, f_n(x) = 0$ for $x < n, f_n(x) = 1$ if $x \geq n$.

**Proof.** Consider the “bad sets”:

$$A = E\{f = \pm\infty\}; \quad A_n = E\{f_n = \pm\infty\}; \quad B = E\{f_n \not\to f\}.$$  

Then $Q = A \cup (\cup_{n \geq 1} A_n) \cup B$ has measure zero. Fixing $\sigma > 0$, let

$$R_n(\sigma) = \bigcup_{k=n}^{\infty} E_k(\sigma), \quad M = \bigcap_{n=1}^{\infty} R_n(\sigma),$$  

a decreasing intersection. Since $m(E) < \infty$, we have $m(M) = \lim_n m(R_n(\sigma))$.

But it is easy to see that $M \subset Q$. So $\lim_n m(R_n(\sigma)) = 0$, which is even stronger than the claim, since $E_n(\sigma) \subset R_n(\sigma)$. This concludes the proof.

A small extension of the proof leads to a stronger result:

**Egorov’s theorem.** Let $f_n, f : E \to R$, where $m(E) < \infty$. Then for any $\delta > 0$ we may find $F \subset E$ measurable with $m(F) \leq \delta$, so that $f_n \to f$ uniformly on $E \setminus F$.

**Proof.** We showed earlier that, for any $\sigma > 0$, $m(R_n(\sigma)) \to 0$. Let $(\sigma_i)_{i \geq 1}$ be any decreasing sequence of positive numbers converging to zero. Given $\delta > 0$, we find $n_i$ so that:

$$m(R_{n_i}(\sigma_i)) < \frac{\delta}{2^i} \quad \forall i \geq 1.$$  

Then letting

$$F = \bigcup_{i=1}^{\infty} R_{n_i}(\sigma_i), \quad m(F) \leq \sum_{i=1}^{\infty} m(R_{n_i}(\sigma_i)) \leq \delta,$$  

it is easy to see that $f_n \to f$ uniformly in $E \setminus F$. Indeed given $\epsilon > 0$ choose $i_0 \geq 1$ so that $\sigma_{i_0} < \epsilon$. Then if $k \geq n_{i_0}$ and $x \in E \setminus F$, one verifies easily that:

$$|f_k(x) - f(x)| \leq \sigma_{i_0} < \epsilon.$$
Proposition 1 motivates the following definition.

Definition. Let $f_n, f : E \to \overline{R}$ be measurable and a.e. finite. We say $f_n$ converges to $f$ in measure if for all $\sigma > 0 \lim_{n \to \infty} m(E_n(\sigma)) = 0$, where $E_n(\sigma) = \{x \in E; |f_n(x) - f(x)| \geq \sigma\}$.

Remark. The limit in measure of a sequence $(f_n)$ is not unique, but any two limits coincide a.e. ([Natanson, p.97]).

We showed in Proposition 1 that pointwise convergence implies convergence in measure (for functions defined on a set of finite measure). Conversely, if $f_n \to f$ in measure, then a subsequence of $(f_n)$ converges to $f$ pointwise a.e.

Proposition 2. Let $f_n, f : E \to R$, where $m(E) < \infty$. Assume $f_n \to f$ in measure. Then a subsequence $(f_{n_i})$ converges to $f$ a.e. in $E$.

Proof. With notations as before, we have $m(E_n(\sigma)) \to 0$. Let $\sigma_i > 0$ be a decreasing sequence with limit zero. For each $i \geq 1$ we may find $n_i \geq 1$ so that:

$$m(E_{n_i}(\sigma_i)) \leq \frac{1}{2^i},$$

and hence $m(R_k) \leq \frac{1}{2^k}$, where $R_k = \bigcup_{i=k}^{\infty} E_{n_i}(\sigma_i)$.

Thus, defining:

$$N = \bigcap_{k=1}^{\infty} R_k,$$

the decreasing intersection property implies $m(N) = 0$. We claim that $f_{n_i}(x) \to f(x)$ for $x \in E \setminus N$.

To see this, let $\epsilon > 0$ be given, and let $x \in E \setminus N$. This means for some $k \geq 1$ we have: $x \in E \setminus R_k$, so for all $i \geq k$: $x \in E \setminus E_{n_i}(\sigma_i)$. Choosing $i_0 \geq k$ so that $\sigma_{i_0} < \epsilon$, we have for all $i \geq i_0$: $|f_{n_i}(x) - f(x)| < \epsilon$, as claimed.

The next result says that any given any measurable function $f$ we may find a closed subset of its domain whose complement has arbitrarily small measure, so that the restriction of $f$ to this closed set is continuous.

Luzin’s theorem. Let $f : E \to R$ be a measurable function. Then for any $\delta > 0$ we may find $F \subset E$ closed so that the restriction $f|_F$ is continuous on $F$ and $m(E \setminus F) \leq \delta$.

Proof. (i) Assume first $m(E) < \infty$. For each integer $k \geq 1$, we let $\{I_{k,n}\}_{n \geq 1}$ denote the partition of $R$ into countably many intervals (left-closed, right-open) of length $1/k$, and consider the partition of $E$ by their
preimages, \[ E = \bigcup_{n=1}^{\infty} E_{k,n}, \quad E_{k,n} = f^{-1}(I_{k,n}). \]

For each \( n \geq 1 \) we may find \( F_{k,n} \subset E_{k,n} \) compact, so that \( m(E_{k,n} \setminus F_{k,n}) < \frac{\delta}{2^{k+n+1}} \), in particular:

\[ m(E \setminus \bigcup_{n=1}^{\infty} F_{k,n}) \leq \frac{\delta}{2k+1}, \quad m(E \setminus F_k) \leq \frac{\delta}{2k}, \quad \text{where} \quad F_k = \bigcup_{n=1}^{N_k} F_{k,n}, \]

for some \( N_k \geq 1 \) sufficiently large. Note the \( F_k \) are closed sets, hence their intersection \( F = \cap_{k \geq 1} F_k \) is also closed, and its complement in \( E \) has measure estimated by:

\[ m(E \setminus F) = m(\bigcup_{k \geq 1} E \setminus F_k) \leq \delta. \]

Now define \( \phi_k : F_k \to R \) by:

\[ \phi_k(x) = y_{k,n} \quad \text{for} \quad x \in F_{k,n}, \]

where \( y_{k,n} \in I_{k,n} \) is the left endpoint of the interval \( I_{k,n} \). This is well-defined, since the \( F_{k,n} \) for different \( n \) are disjoint. Further, since \( \phi_k \) is constant on disjoint closed sets, it is continuous in \( F_k \).

It is easy to see we have, for each \( x \in F_k \) (in particular, for each \( x \in F \)):

\[ |\phi_k(x) - f(x)| \leq \frac{1}{k}. \]

This shows \( \phi_k \to f \) uniformly in \( F \), hence \( f \) is continuous when restricted to \( F \), as claimed in the statement of the theorem.

(ii) To extend this to the case when \( m(E) \) is not finite, consider the partition of \( R^n \) into countably many cubes \( (Q_j)_{j \geq 1} \), say of side length one. We may apply part (i) to conclude the existence of \( F_j \subset E \cap Q_j \) with:

\[ m((E \cap Q_j) \setminus F_j) \leq \frac{\delta}{2^j}, \quad f|_{F_j} \text{ continuous}. \]

Since the family of cubes \( \{Q_j\} \) is locally finite, the countable union of closed sets \( F = \cup_{j \geq 1} F_j \) is also closed. (Check this.) Thus \( f|_F \) is continuous and we estimate the measure of \( E \setminus F \) by:

\[ m(E \setminus F) \leq \sum_{j=1}^{\infty} m((E \cap Q_j) \setminus F_j) \leq \delta, \]
as we wished to show.

**Problem 3.** Prove the converse: if $f : E \to R$ is a function with the property that for any $\delta > 0$ one may find a closed set $F \subset E$ so that the restriction $f|_F$ is continuous and $m(E \setminus F) < \delta$, then $f$ is measurable.

**Corollary 1.** If $f : E \to R$ is measurable, for any $\delta > 0$ we may find $g_{\delta} : E \to R$ continuous in $E$, so that $m(\{x \in E; f(x) \neq g_{\delta}(x)\}) < \delta$. If $|f(x)| \leq K$ in $E$, then also $|g_{\delta}(x)| \leq K$ in $E$.

This follows from Tietze’s extension theorem in Topology (which says we can extend continuous functions defined on a closed subset to continuous functions on the whole space, without increasing its sup norm): extending the function $f$ from the closed set $F$ given by Luzin’s theorem to all of $E$ yields $g_{\delta}$.

**Corollary 2.** Let $f : E \to R$ be measurable. Then there exists a sequence $f_n : E \to R$ of functions continuous in $E$ so that $f_n \to f$ a.e. in $E$.

**Proof.** Assume first $m(E) < \infty$. Letting $\delta_n$ be any sequence converging to 0 and considering the functions $f_n = g_{\delta_n}$ (continuous in $E$) given by Corollary 1, we see that $f_n \to f$ in measure. Thus by Proposition 2 a subsequence $f_{n_j}$ converges to $f$ a.e. in $E$.

It is easy to extend this to the case $m(E) = \infty$ (left to the reader.)