RIEMANN-STIELTJES INTEGRALS.

Given bounded functions $f, g : [a,b] \to R$ and a partition $P = \{x_0 = a, x_1, \ldots, x_n = b\}$ of $[a,b]$, and a 'tagging' $t_k \in [x_{k-1}, x_k]$, form the *Riemann-Stieltjes sum* for the tagged partition P^*:

$$S_g(f, P^*) = \sum_{k=1}^{n} f(t_k)[g(x_k) - g(x_{k-1})].$$

If the limit of these sums (as the mesh $||P|| \to 0$) exists, we set:

$$\int_a^b f dg = \lim_{||P|| \to 0} S_g(f, P^*).$$

Theorem 1. If $f \in C[a,b]$ and $g \in BV[a,b]$, $\int_a^b f dg$ exists.

Proof. If is enough to assume g is increasing (why?) Given a partition P, form the Darboux-type sums:

$$S_g^-(f,P) = \sum_{k=1}^{n} (\inf_{I_k} f)[g(x_k) - g(x_{k-1})], \quad S_g^+(f,P) = \sum_{k=1}^{n} (\sup_{I_k} f)[g(x_k) - g(x_{k-1})],$$

where $I_k = [x_{k-1}, x_k]$. Clearly for any 'tagging' $\{t_k\}$, we have:

$$S_g^-(f,P) \leq S_g(f,P^*) \leq S_g^+(f,P^*).$$

Given $\epsilon > 0$, from uniform continuity of f we may find $\delta > 0$ so that the oscillation $osc_{I_k} f$ (sup minus inf) is smaller than ϵ, over each interval of a partition with mesh less than δ. Let $I = \sup_P S_g^-(f,P)$ (sup over all partitions). We have:

$$|S_g(f,P^*) - I| \leq S_g^+(f,P) - S_g^-(f,P) \leq \sum_k (osc_{I_k} f)[g(x_k) - g(x_{k-1})] \leq \epsilon(g(b) - g(a)).$$

This shows $\int_a^b f dg$ exists, and equals $\sup_P S_g^-(f,P)$ (or $\inf_P S_g^+(f,P)$), if g is increasing.

Corollary 1. Under the same hypotheses, we have: $|\int_a^b f dg| \leq ||f||_{sup} V_a(g)$.

It suffices to observe that, for any tagged partition P^*:

$$|S_g(f,P^*)| \leq \sum_{k=1}^{n} |f(t_k)||g(x_k) - g(x_{k-1}| \leq ||f||_{\sup} V_{ab}(g).$$

Corollary 2. If $f_n \in C[a,b]$ converge uniformly to f, and $g \in BV[a,b]$, we have:

$$\int_a^b f_n dg \to \int_a^b f dg.$$

This is clear, since from Corollary 1:

$$|\int_a^b f_n dg - \int_a^b f dg| \leq ||f_n - f||_{\sup} V_{ab}(g) \to 0.$$

Exercise: Show that $\int_a^b 1 dg = g(b) - g(a)$, if $g \in BV[a,b]$.

We turn to two evaluation theorems under simple hypotheses:

Theorem 2. If $f \in C[a,b]$ and g is differentiable on (a,b) with derivative integrable on $[a,b]$ (in particular bounded), we have:

$$\int_a^b f dg = \int_a^b f g'.$$

Note that both integrals (Riemann-Stieltjes on the left, Riemann on the right) are known to exist, under the hypotheses of the theorem.

Proof. Given a partition $P = \{x_0 = a, x_1, \ldots, x_n = b\}$, by the Mean Value Theorem we have for some $t_k \in (x_{k-1}, x_k), k \geq 1$:

$$f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}).$$

Thus we have the equality:

$$S_g(f, P^*) = S_(fg', P^*).$$

(Riemann-Stieltjes sum on the left, Riemann sum on the right.) Letting the mesh $||P|| \to 0$, we have the result.

Theorem 3. Suppose $g \in BV[a,b]$, snd let $\{c_0 = a, c_1, \ldots c_m, c_{m+1} = b\}$ be a partition of $[a,b]$, with the property that g is constant in each open interval $(c_k, c_{k+1}), k = 0, \ldots, m$. Then, for each $f \in C[a,b]$:

$$\int_a^b f dg = f(a)[g(a+) - g(a)] + \sum_{k=1}^m f(c_k)[g(c_k+) - g(c_k-)] + f(b)[g(b) - g(b-)].$$

Proof. Since $g \in BV[c_k, c_{k+1}]$ for each k, we may write:

$$\int_a^b f dg = \sum_{k=0}^m \int_{c_k}^{c_{k+1}} f dg.$$

Fix k, and consider an arbitrary tagged partition $P_k = \{x_0 = c_k, x_1, \dots, x_n = c_{k+1}\}, t_j \in [x_j, x_{j+1}]$ of $[c_k, c_{k+1}]$. Forming the corresponding Riemann-Stieltjes sum, we notice that the only non-vanishing terms are the first and last ones, giving:

$$S_g(f_{|[c_k,c_{k+1}]}, P_k^*) = f(c_k)[g(c_k+) - g(c_k)] + f(c_{k+1})[g(c_{k+1}) - g(c_{k+1}-).]$$

Adding over k, we find that the interior terms involving $g(c_k)$ cancel, and obtain the result claimed.

Remark. Thus we see that in case g is a 'step function', the values of g at the interior discontinuities $c_k \in (a, b)$ don't affect the value of the Riemann-Stieltjes integral (when f is continuous); but the values of g at the endpoints do.(For instance, if $g = 0$ in $[a, b)$, then the Riemann-Stieltjes integral equals $f(b)g(b)$.)

In fact this is true for any $g \in BV[a, b]$, not necessarily a 'step function': the values taken by g at *interior* points of discontinuity don't affect the values of Riemann-Stieltjes integrals of continuous functions with g as 'integrator'. Clearly it suffices to show the following.

Proposition. Suppose $g \in BV[a, b]$ is zero on $[a, b]$, except at countably many points $x_i \in (a, b)$, where it takes the value $p_i \neq 0$. Then $\int_a^b f dg = 0$, for any $f \in C[a, b]$.

Proof. Exercise. Here is an outline: first prove it in the case of finitely many x_i. (This is easy: consider the case of just one point first; the proof use the fact it is an *interior* point, hence there exists a partition of small mesh avoiding it.) Since g has bounded variation,

$$V_{ab}(g) = \sum_i |p_i| < \infty.$$

Given $\epsilon > 0$, choose N so that $\sum_{k=N+1}^{\infty} |p_k| < \epsilon$, and write $g = g_1 + g_2$, where $g_1 = 0$ except at $x_1, \dots, x_N$ (and $g_2 = 0$ except on $\{x_k\}_{k=N+1}^{\infty}$). Then $\int f dg_1 = 0$ for any f, while for any tagged partition:

$$|\sum_{k=1}^{n} f(t_k)[g_2(x_k) - g_2(x_{k-1})| \leq 2||f||_{sup} \sum_{k=N+1}^{\infty} |p_k| < 2||f||_{\infty}\epsilon.$$

This naturally leads to the question: when do two functions $g_1, g_2 \in BV[a, b]$ lead to equal values of Riemann-Stieljes integrals of any continuous function? That is, how sensitive is the value of R-S integrals to the values of g at the endpoints. Clearly we may always subtract a constant, and thus 'normalize' the value of g at a to be zero. Given $g \in BV[a,b]$, define its normalization $\tilde{g} \in BV[a,b]$ by:

$$\tilde{g}(a) = 0; \quad \tilde{g}(t) = g(t-) - g(a) \text{ for } t \in (a,b); \quad \tilde{g}(b) = g(b) - g(a).$$

Thus $\tilde{g}$ is left-continuous on (a,b). Clearly g and $\tilde{g}$ are equivalent, as integrators of continuous functions. The converse is true: if two functions $g_1, g_2 \in BV[a,b]$ have the same effect as integrators of continuous functions, their normalizations coincide: $\tilde{g_1} = \tilde{g_2}$. Equivalently, if g_1, g_2 are both left-continuous in (a,b) and zero at a (and give the same results as integrators of continuous functions), then $g_1(b) = g_2(b)$. *Can you prove this?* Consider what happens if you take $f \equiv 1$.

Recall *Helly's First Theorem* says that if an infinite family $\mathcal{F}$ of functions in $[a,b]$ is uniformly bounded, of uniformly bounded variation, then a sequence of functions in $\mathcal{F}$ converges pointwise on $[a,b]$ to a function of bounded variation.

Theorem 4. (*Helly's second theorem.*)Let g_n be a sequence in $BV[a,b]$, with $V_{ab}(g_n) \leq K$. Assume $g_n \to g$ pointwise on $[a,b]$. (By Helly's First Theorem, we know $g \in BV[a,b]$.) Then, for any $f \in C[a,b]$:

$$\int_a^b f dg_n \to \int_a^b f dg.$$

Proof. Given $\epsilon > 0$, take a partition $\{x_k\}_{k=0}^m$ such that $osc_{I_k} f < \epsilon$, for all k. Then:

$$\int_a^b f dg = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} (f(x) - f(x_k)) dg + \sum_{k=0}^{m-1} f(x_k)(g(x_{k+1}) - g(x_k)).$$

The first sum is bounded above by $\epsilon V_{ab}(g)$. Thus for some real number C with $|C| \leq 1$, we have:

$$\int_a^b f dg = \sum_{k=0}^{m-1} f(x_k)(g(x_{k+1}) - g(x_k)) + C\epsilon V_{ab}(g).$$

And, likewise, for each n:

$$\int_a^b f dg_n = \sum_{k=0}^{m-1} f(x_k)(g_n(x_{k+1}) - g_n(x_k)) + C\epsilon V_{ab}(g_n).$$

On the other hand, we may find N so that, for $n \geq N$:

$$|\sum_{k=0}^{m-1} f(x_k)(g_n(x_{k+1}) - g_n(x_k)) - \sum_{k=0}^{m-1} f(x_k)(g(x_{k+1}) - g(x_k))| < \epsilon.$$

We conclude, for $n \geq N$:

$$|\int_a^b f dg - \int_a^b f dg_n| \leq \epsilon + 2CK\epsilon \leq (1 + 2K)\epsilon,$$

proving the claim.

The main application of Riemann-Stieltjes integration is that it leads to a concrete representation of all bounded linear functionals Φ on $C[a, b]$. (This means $\Phi : C[a, b] \to R$ is linear, with $|\Phi(f)| \leq K||f||_{sup}$ (for some $K > 0$ independent of f.) This is the content of *Riesz's Theorem.*

Theorem 5. (Riesz.) If Φ is a bounded linear functional on $C[a, b]$ (with the sup norm), there exists a function $g \in BV[a, b]$ so that: $\Phi(f) = \int_a^b f dg$, for all $f \in C[a, b]$.

Proof. (Outline.) Recall Bernstein's theorem on polynomial approximation (*see the Appendix to this handout.*) Consider the linear operator on $C[0, 1]$ (for each $n \geq 1$):

$$B_n[f](x) = \sum_{k=0}^{n} f(\frac{k}{n}) C_n^k x^k (1 - x)^{n-k}.$$

Then $B_n[f] \to f$ uniformly in $[0, 1]$. Thus $\Phi(B_n[f]) \to \Phi(f)$.

Claim. We may find $g_n \in BV[0, 1]$, increasing step functions satisfying $V_{01}(g_n) \leq K$ and $||g_n||_{sup} \leq K$, so that for all $f \in C[0, 1]$:

$$\int_0^1 f dg_n = \sum_{k=0}^{n} f(\frac{k}{n}) \Phi(C_n^k x^k (1 - x)^{n-k}).$$

Given the claim, by Helly's first theorem we may find a subsequence so that $g_n \to g \in BV[0, 1]$, pointwise in $[0, 1]$. Thus by Helly's second theorem:

$$\int_0^1 f dg_n \to \int_0^1 f dg;$$

while:

$$\int_0^1 f dg_n = \Phi(B_n(f)) \to \Phi(f).$$

Thus we conclude $\Phi(f) = \int_0^1 f dg$, as desired.

Proof of claim. Define $g_n \in BV[0,1]$ by: $g_n(0) = 0$,

$$g_n(x) = \Phi(C_n^0(1-x)^n), \quad 0 < x < 1/n,$$

$$g_n(x) = \Phi(C_n^0(1-x)^n) + \Phi(C_n^1 x(1-x)^{n-1}), \quad \frac{1}{n} \leq x < \frac{2}{n}, \ldots,$$

$$g_n(x) = \sum_{k=0}^{n-1} \Phi(C_n^k x^k (1-x)^{n-k}), \quad \frac{n-1}{n} \leq x < 1,$$

$$g_n(1) = \sum_{k=0}^{n} \Phi(C_n^k x^k (1-x)^{n-k}).$$

Now $\sum_{k=0}^n C_n^k x^k (1-x)^{n-k} = 1$ implies that, for any choice of signs $\epsilon_k = \pm 1$, we have $|\sum_{k=0}^n C_n^k \epsilon_k x^k (1-x)^{n-k})| \leq 1$, hence:

$$|\sum_{k=0}^{n} \epsilon_k \Phi(C_n^k x^k (1-x)^{n-k})| \leq K.$$

For a suitable choice of the signs ϵ_k, each term in the sum is positive, so we have:

$$V_{01}(g_n) = \sum_{k=0}^{n} |\Phi(C_n^k x^k (1-x)^{n-k})| \leq K.$$

This implies $||g_n||_{sup} \leq K$ as well, since $g_n(0) = 0$.

DIFFERENTIAL FORMS AND LINE INTEGRALS.

Line integrals can be thought of as a vector-valued version of Riemann-Stieltjes integrals, along continuous, rectifiable curves. An important property of line integrals is invariance under reparametrization of the curve; so we consider this first for Riemann-Stieltjes integrals.

Theorem 6. Let $f \in C[c,d], g \in BV[c,d]$. Let $\phi : [a,b] \to [c,d]$ be an increasing homeomorphism. Then $g \circ \phi \in BV[a,b]$ (in fact $V_{ab}(g \circ \phi) = V_{cd}(g)$) and:

$$\int_a^b (f \circ \phi) d(g \circ \phi) = \int_c^d f dg. \quad (Riemann-Stieltjes).$$

Proof. Given a tagged partition of $[a,b]$, $P = \{x_k\}_{k=0}^m$ of $[a,b]$, $t_k \in [x_{k-1}, x_k]$, we define a tagged partition P_ϕ of $[c,d]$ by setting $y_k = \phi(x_k), s_k =$

$\phi(t_k)$. Clearly this defines a bijection of tagged partitions, and it is easy to check that:

$$S_g(f, P_\phi^*) = S_{g\circ\phi}(f \circ \phi, P^*).$$

The only other thing to check is that $(\forall \delta > 0)(\exists \delta_1 > 0)(||P|| < \delta_1 \Rightarrow ||P_\phi|| < \delta)$. This follows easily from the uniform continuity of ϕ. (And similarly for ϕ^{-1}).

In the same way we see that $V_P(g \circ \phi) = V_{P_\phi}(g)$, implying the claims about $g \circ \phi$.

Consistency check. We should see what this says in case g is differentiable (with Riemann-integrable derivative) and ϕ is a C^1 diffeomorphism. Then $g \circ \phi$ is differentiable, with Riemann-integrable derivative. To see this, note $(g \circ \phi)' = (g' \circ \phi)\phi'$, at every point $x \in (a, b)$ such that ϕ is differentiable at x and g is differentiable at $\phi(x)$. Since ϕ is bi-Lipschitz (by the MVT) it takes sets of measure zero in $[a, b]$ to sets of measure zero in $[c, d]$, and vice-versa. Thus the set of such x has full measure in $[a, b]$. Then Theorem 6 says:

$$\int_a^b (f \circ \phi)(g' \circ \phi)\phi' = \int_c^d fg' \quad (\textit{Riemann}), \quad \forall f \in C[a, b].$$

This indeed follows from the Change of Variable Theorem for the Riemann integral, applied to the function fg'.

Change of Variables Theorem. Let $\phi : [a, b] \to [c, d]$ be a C^1 diffeomorphism (increasing). If f is Riemann-integrable in $[c, d]$, then $(f \circ \phi)\phi'$ is Riemann-integrable in $[a, b]$ and:

$$\int_a^b (f \circ \phi)\phi' = \int_c^d f \qquad (\textit{Riemann}).$$

Proof. **Exercise 1.**

Definitions. A C^k vector field on an open set $U \subset R^n$ is a C^k map $X : U \to R^n$. In terms of components in the standard basis $\{e_i\}$ of R^n, we have: $X(x) = \sum_i a^i(x)e_i$, $x \in U$, where each $a^i \in C^k(U)$ $(k \geq 0)$. Denote by χ_U the space of C^k vector fields in U. Note that $fX \in \chi_U$ if $X \in \chi_U$ and $f \in C^k(U)$ is a function.

A C^k one-form in U is a map $\omega : \chi_U \to C^k(U)$ (maps vector fields to functions) which is 'linear over functions', in the sense that $\omega \cdot (fX) = f(\omega \cdot X)$. Denote by dx^j $(j = 1, \ldots, n)$ the one-form which maps each vector field to its jth. component:

$$X = \sum a^i e_i, a^i \in C^k(U) \Rightarrow dx^j \cdot X = a^j.$$

Then each one-form can be written as $\omega = \sum_j b_j dx^j$, for functions $b_j \in C^k(U)$, $J = 1, \ldots, n$. Denote by Ω^1_U the space of C^k one-forms in U. Note that at each $x \in U$, $\omega(x)$ defines a linear functional in R^n –an element of the 'dual space' $(R^n)^*$:

$$\omega = \sum_j b_j dx^j \Rightarrow \omega(x)[v] = \sum_j b_j(x) v^j, \quad v = (v^1, \ldots, v^n) \in R^n.$$

Example. Let $f \in C^{k+1}(U)$. Then its differential $df : U \to (R^n)^*$ defines a one-form of class C^k in U. In components:

$$df = \sum_j \frac{\partial f}{\partial x^j} dx^j, \quad (df \cdot X)(x) = \sum_j \frac{\partial f}{\partial x^j} a^j \text{ if } X = \sum a^i e_i.$$

Line integrals of one-forms along rectifiable paths.

Let $\omega \in \Omega^1_U$ be a one-form of class C^k $(k \geq 0)$. Let $\gamma : [a, b] \to U$ be a rectifiable path in U (continuous, and each component $\gamma^i \in BV[a, b]$.) For a given tagged partition of $[a, b]$ ($t_k \in [x_{k-1}, x_k]$ $(x_0 = a, x_m = b)$, consider the Riemann-Stieltjes-type sum:

$$S_\gamma(\omega, P^*) = \sum_{k=0^m} \omega(\gamma(t_k)) \cdot [\gamma(t_k) - \gamma(t_{k-1})].$$

We say the line integral of ω along γ is defined if the limit of these numbers as the mesh of P tends to zero exists, and set:

$$\int_\gamma \omega = \lim_{||P|| \to 0} S_\gamma(\omega, P^*).$$

In components, we have, if $\omega = \sum_j b_j dx^j$ and $\gamma = (\gamma^1, \ldots, \gamma^n)$:

$$\int_\gamma \omega = \sum_j \int_a^b (b_j \circ \gamma) d\gamma^j \qquad (Riemann - Stieltjes).$$

The following facts follow immediately from the corresponding ones for the Riemann-Stieltjes integral:

(i) If ω is a C^0 one-form in U and γ is a (continuous) rectifiable path in U, the line integral exists, and:

$$|\int_\gamma \omega| \leq ||\omega_{|\gamma(I)}||_{sup} L[\gamma], \quad I = [a,b].$$

(Note $\gamma(I)$ is a compact subset of U.)

(ii) If $\omega = \sum_j b_j dx^j$, $b_j \in C(U)$ and each component γ^j is differentiable, with integrable derivative, then:

$$\int_\gamma \omega = \sum_j \int_a^b (b_j \circ \gamma)(\gamma^j)' \qquad (\textit{Riemann}).$$

(iii). (Invariance under reparametrization). Let $\phi : [a,b] \to [c,d]$ be an increasing homeomorphism. Then

$$\int_{\gamma \circ \phi} \omega = \int_\gamma \omega.$$

In components, with $\omega = \sum_j b_j dx^j$, this reads:

$$\sum_j \int_a^b ((b_j \circ \gamma) \circ \phi) d(\gamma^j \circ \phi) = \sum_j \int_c^d (b_j \circ \gamma) d\gamma^j,$$

which follows from the reparametrization invariance of Riemann-Stieltjes integrals.

(iv) (Continuity) (a) If $\omega_n \in \Omega^1_U$ (continuous), γ is a rectifiable (in part. continuous) curve in U, $\omega_n \to \omega$ uniformly in $\gamma([a,b])$ (a compact subset of U), then:

$$\int_\gamma \omega_n \to \int_\gamma \omega.$$

(b) Suppose γ_n, γ are rectifiable curves in $U \subset R^n$ (open) and $\gamma_n \to \gamma$ (in BV norm for each component.) Then, for any $\omega \in \Omega^1_U$ (continuous), we have:

$$\int_{\gamma_n} \omega \to \int_\gamma \omega.$$

Line integrals of exact forms. A one-form ω of class C^k is *exact* in U if $\omega = df$, for some function $f \in C^{k+1}(U)$. (f is referred to as a "potential" for ω in U; clearly $f + C$ is also a potential, for any constant C.)

Proposition. Let $\omega = df$, where $f \in C^1(U)$; let $\gamma : [a,b] \to U$ be a rectifiable curve. Then:

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a)).$$

Proof. Under the additional hypothesis that γ is differentiable, with (Riemann) integrable derivative, this is easy:

$$\int_\gamma \omega = \sum_i \int_a^b (\frac{\partial f}{\partial x^i} \circ \gamma)(\gamma^i)'$$

$$= \int_a^b (f \circ \gamma)' = f(\gamma(b)) - f(\gamma(a)),$$

using the Chain Rule and FTC2.

With γ assumed only rectifiable, there is more work to do. Let $V \subset U$ be an open subset with compact closure $\bar{V} \subset U$, containing the image of γ: $\gamma[a,b] \subset V$. Given $\epsilon > 0$, let $R > 0$ be such that $||df(z) - df(w)|| < \eta < \epsilon/L[\gamma]$ if $||z - w|| < R$ and $z, w \in \bar{V}$ (uniform continuity of df.) Then consider balls $B_p(r)$ centered at each $p \in \gamma[a,b]$, contained in V and with radius smaller than $R/2$. Such balls cover the compact set $\gamma[a,b]$, and we can let $\lambda > 0$ be a Lebesgue number for this cover. Thus if $||\gamma(x_k) - \gamma(x_{k-1})|| < \lambda$, then $\gamma(x_k), \gamma(x_{k-1}) \in B \subset V$, the same ball of radius less than $R/2$, contained in V. In particular, the line segment $[\gamma(x_{k-1}), \gamma(x_k)]$ is also contained in B.

Thus we may choose $\delta > 0$ small enough that any partition $\{x_k\}_{k=0}^m$ with mesh smaller than δ satisfies $||\gamma(x_k) - \gamma(x_{k-1})|| < \lambda$, as well as:

$$|\int_\gamma \omega - \sum_k df(\gamma(t_k))[\gamma(x_k) - \gamma(x_{k-1})]| < \epsilon,$$

for any tagging $\{t_k\}$ (in particular for $t_k = x_{k-1}$.) By the MVT, we may find, for each $k = 0, \dots, n$, $\theta_k \in (0,1)$ so that:

$$f(\gamma(x_k)) - f(\gamma(x_{k-1})) = df(z_k)[\gamma(x_k) - \gamma(x_{k-1})], \quad z_k = \gamma(x_{k-1}) + \theta_k(\gamma(x_k) - \gamma(x_{k-1})).$$

Since:

$$|\sum_k (df(z_k) - df(\gamma(x_{k-1}))[\gamma(x_k) - \gamma(x_{k-1})]| \le \eta \sum_k ||\gamma(x_k) - \gamma(x_{k-1})|| \le \eta L[\gamma] < \epsilon$$

we conclude:

$$|\int_\gamma \omega - \sum_k df(z_k)[\gamma(x_k) - \gamma(x_{k-1})]| < 2\epsilon.$$

And then we just have to observe that the sum equals $f(\gamma(b)) - f(\gamma(a))$.

Converse. Let $U \subset R^n$ be a connected open set, $\omega \in \Omega^1_U$ a one-form with continuous coefficients. Suppose the line integrals of ω along rectifiable curves in U depend only on the endpoints of the curve. Then ω is exact in U.

Proof. Fix $p \in U$, and for $x \in U$ define:

$$f(x) = \int_{\gamma_{px}} \omega,$$

where γ_{px} is any rectifiable curve in U from p to x. (See the exercise below.) Then:

$$\frac{\partial f}{\partial x_i}(x) = \frac{d}{dt}[f(x + te_i) - f(x)]_{|t=0} = \frac{d}{dt}\int_{\gamma_{xx+te_i}} \omega$$

$$= \frac{d}{dt}\int_0^t \omega(x + se_i)[e_i]ds_{|t=0} = b_i(x), \quad \omega(x) = \sum_i b_i(x)dx^i$$

as we wished to show.

Exercise 2. Let $U \subset R^n$ be a connected open set. (Recall this means any two points of U may be joined by a continuous curve in U.) Show any two points of U may be joined by a *rectifiable* curve in U. (*Hint:* in fact, any two points may be joined by a *polygonal* curve in U–a concatenation of finitely many line segments. See the proposition just proved.)

Definitions. A 2-form in U is a map $\alpha : \chi_U \times \chi_U \to C^k(U)$ $(k \geq 0)$, taking pairs of vector fields to functions, linear over functions and skew-symmetric:

$$\alpha(fX, Y) = f\alpha(X, Y), \quad \alpha(Y, X) = -\alpha(X, Y), \qquad \text{so } \alpha(X, X) = 0, \forall X, Y \in \chi_U, f \in C^k(U).$$

Basis two-forms: For $i < j, i, j \in \{1, \ldots, n\}$, consider the two-forms defined by:

$$\alpha^{ij}(X, Y) = a^i b^j - a^j b^i, \text{ if } a = (a^1, \ldots, a^n), Y = (b^1, \ldots, b^n).$$

The usual notation for α^{ij} is $dx^i \wedge dx^j$ (see the next example for the reason.) Any $\alpha \in \Omega^2_U$ (the space of two-forms in U) may be written as a linear combination:

$$\alpha = \sum_{i<j} a_{ij} dx^i \wedge dx^j, \quad a_{ij} \in C^k(U); \quad a_{ij} = \alpha(e_i, e_j) \text{ for } i < j >$$

Wedge product. Given $\alpha, \beta \in \Omega^1_U$, define their wedge product $\alpha \wedge \beta \in \Omega^2_U$ by:

$$(\alpha \wedge \beta)(X, Y) = (\alpha \cdot X)(\beta \cdot Y) - (\alpha \cdot Y)(\beta \cdot X).$$

Exterior differential.
Given $\omega = \sum_i b_i dx^i \in \Omega^1_U$, define:

$$d\omega = \sum_{i<j} (\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}) dx^i \wedge dx^j.$$

Exercise 3. Show that:
$d(f\omega) = df \wedge \omega + f d\omega, \quad \forall f \in C^1(U), \omega \in \Omega^1_U$ (of class C^1).

A one-form ω is *closed* if $d\omega = 0$. An easy calculation shows that, for any $f \in C^2$, we have: $d(df) = 0$. Thus *any 1-form exact in U is closed in U*. The converse is false in general.

Example. (Angle one-form in the plane.) Consider the one-form in $U = R^2 \setminus \{0\}$ (with coordinates (x, y):

$$\theta = \frac{xdy - ydx}{x^2 + y^2}.$$

It is easy to check that $d\theta = 0$, so θ is closed in U. But it is not exact, since $\int_\gamma \theta = 2\pi$, if γ is the unit circle (traversed once in the counterclockwise direction.). On the other hand, with another choice of domain, θ may be exact. For example, if $V = R^2 \setminus \{(x, 0); x \leq 0\}$, the function $f(x, y) = \arctan(y/x) \in C^1(V)$ (where we take arctan with values in $(-\pi, \pi)$) satisfies $df = \theta$ in V, as is easily checked.

Proposition. Any closed 1-form $\omega \in \Omega^1_U$ (with C^1 coefficients) is *locally exact*: for any point $p \in U$, we bay find a sufficiently small open ball $B = B_r(p) \subset U$ so that ω is exact in B.

Proof. Consider the line integral of ω along the line segment σ_x from p to $x \in B$: choosing coordinates so that p is the origin, and letting $\omega = \sum_i b_i dx^i$, we define the function in B:

$$f(x) = \int_{\sigma_x} \omega = \sum_i \int_0^1 b_i(tx)x^i dt.$$

We compute the partial derivative, using the hypothesis and differentiating under the integral sign:

$$\frac{\partial f}{\partial x^j}(x) = \int_0^1 \{\sum_i (\frac{\partial b_i}{\partial x^j}(tx)tx^i) + b_j(tx)\}dt$$

$$= \int_0^1 \{\sum_i (\frac{\partial b_j}{\partial x^i}(tx)tx^i) + b_j(tx)\}dt = \int_0^1 \frac{d}{dt}(tb_j(tx)) = b_j(x),$$

as claimed.

This is a good occasion to state and prove a useful result on 'differentiating under the integral':

Theorem. Let $f : U \times [a,b] \to R$, $U \subset R^n$, have the properties:
(i) $t \mapsto f(x,t)$ is (Riemann) integrable in $[a,b]$, for any $x \in U$;
(ii) $\frac{\partial f}{\partial x^i}(x,t)$ exists and is continuous in $U \times [a,b]$.
Then if $\phi(x) = \int_a^b f(x,t)dt$ for $x \in U$, we have:

$$\frac{\partial \phi}{\partial x^i}(x) = \int_a^b \frac{\partial f}{\partial x^i}(x,t)dt.$$

Thus $\phi \in C^1(U)$.

Proof. Using the Mean Value Theorem, for each $x \in U, s \neq 0$ we may find $\theta \in (0,1)$ so that:

$$\frac{1}{s}[\phi(x+se_i) - \phi(x)] - \int_a^b \frac{\partial f}{\partial x^i}dt$$

$$= \int_a^b [\frac{1}{s}(f(x+se_i,t) - f(x,t)) - \frac{\partial f}{\partial x^i}(x,t)]dt$$

$$= \int_a^b [\frac{\partial f}{\partial x^i}(x+\theta se_i,t) - \frac{\partial f}{\partial x^i}(x,t)]dt.$$

For any $\epsilon > 0$, we may find $\delta > 0$ so that the integrand is bounded by $\epsilon/(b-a)$ (in absolute value) if $|s| < \delta$ (by uniform continuity). Since the limit of the left-hand side as $s \to 0$ is $\frac{\partial \phi}{\partial x^i} - \int_a^b \frac{\partial f}{\partial x^i} dt$, this proves the claim.

Exercise 4. Let ω be a one-form of class C^1 in $U \subset R^n$. Assume that for any closed rectifiable curve γ in U, the line integral $\int_\gamma \omega$ is a rational number. Show that ω is closed in U.

Hint. It is enough to show ω is locally exact: given any point $p \in U$, there exists an open ball B with center p so that ω is exact in B. Assume $p = 0$, and suppose $\int_\gamma \omega \neq 0$, for some rectifiable closed curve γ with image in B. For $\lambda \in [0,1]$, consider the closed curve γ_λ (image of γ under the map of R^n $x \mapsto \lambda x$). Note that the line integral of ω along γ_λ depends continuously on λ. Now use the intermediate value theorem.

Exercise 5. Suppose a 1-form ω, of class C^1 and closed in $R^2 \setminus \{0\}$, has the property that its coefficient functions are bounded in a neighborhood of 0. Show that ω is exact in $R^2 \setminus \{0\}$.

Hint: Changing the proof of the theorem slightly, define, for each $\epsilon > 0$ and $x \in R^2 \setminus \{0\}$:

$$f_\epsilon(x) = \sum_i \int_\epsilon^1 b_i(tx) x^i dt, \quad \omega = \sum_i b_i dx^i.$$

Show that:

$$\frac{\partial f_\epsilon}{\partial x^j}(x) = b_j(x) - \epsilon b_j(\epsilon x),$$

and that this implies:

$$df_\epsilon \to \omega$$

uniformly in each punctured ball centered at 0, $B_R^* \subset R^2 \setminus \{0\}$. Explain why this implies f_ϵ converges to a function f in $R^2 \setminus \{0\}$ (uniformly in each such ball B_R^*), where $df = \omega$.

APPENDIX: WEIERSTRASS APPROXIMATION ON $[0,1]$.

It is a remarkable fact that, for uniform approximation by polynomials in the unit interval $[0,1]$, there is an explicit procedure (via 'Bernstein polynomials') that amounts almost to "a formula".

Denote by $C_j^n = C_{n-j}^n$ the binomial coefficient: $C_j^n = \frac{n!}{j!(n-j)!}, 0 \leq j \leq n$. As we learn in high school:

$$\sum_{j=0}^{n} C_j^b x^j (1-x)^{n-j} = (x+1-x)^n = 1, \quad x \in [0,1].$$

We use these terms as coefficients and, for each $n \geq 1$, 'sample' the function $f \in C[0,1]$ at equidistant points to define the polynomial $B_n[f](x)$:

$$B_n[f](x) = \sum_{j=0}^{n} f(\frac{j}{n}) C_j^n x^j (1-x)^{n-j}.$$

Bernstein's Theorem: $B_n[f] \to f$ uniformly in $[0,1]$.

Proof. First note that $B_n[f](0) = f(0), B_n[f](1) = f(1)$. Then, letting $q_{nj}(x) = C_j^n x^j (1-x)^{n-j}$, we have:

$$\sum_{j=0}^{n} q_{nj}(x) \equiv 1 \Rightarrow |f(x) - B_n[f](x)| \leq \sum_{j=0}^{n} |f(x) - f(\frac{j}{n})| q_{nj}(x).$$

By uniform continuity of f, given $\epsilon > 0$ we may find $\delta > 0$ (depending only on ϵ and f) so that $|f(x) - f(\frac{j}{n})| < \epsilon$ whenever $|x - \frac{j}{n}| < \delta$. So for each $x \in [0,1]$ we split the points $\frac{j}{n}$ in $[0,1]$ into two sets:

$$N_1 = \{j = 1, \ldots, n; |x - \frac{j}{n}| < \delta\}, \quad N_2 = \{j = 1, \ldots, n; |x - \frac{j}{n}| \geq \delta\}.$$

The sum over N_1 is easy to estimate:

$$\sum_{j \in N_1} |f(x) - f(\frac{j}{n})(x)| q_{nj}(x) < \epsilon \sum_{j=0}^{n} q_{nj}(x) = \epsilon.$$

To estimate the other sum, we need a lemma.

Lemma. $\sum_{j=0}^{n} q_{nj}(x)(x - \frac{j}{n})^2 = \frac{x(1-x)}{n} \leq \frac{1}{4n}$.

Assuming the lemma, with $|f(x)| \leq M$ in $[0,1]$ we have:

$$\sum_{j\in N_2} |f(x) - f(\frac{j}{n})(x)| q_{nj}(x) \leq 2M \sum_{j=0}^{n} q_{nj}(x) \frac{(x-\frac{j}{n})^2}{\delta^2} \leq \frac{M}{2n\delta^2} < \epsilon,$$

provided $n > M/2\delta^2$. This concludes the proof.

Proof of Lemma. Expanding $(x-\frac{j}{n})^2$, we see it is enough to compute:

$$B_n[1](x) = \sum_{j=0}^{n-1} q_{nj}(x) = 1;$$

using $(j/n)C_j^n = C_{j-1}^{n-1}$:

$$B_n[x](x) = \sum_{j=0}^{n} q_{nj}(x)\frac{j}{n} = x\sum_{j=1}^{n} C_{j-1}^{n-1} x^{j-1}(1-x)^{(n-1)-(j-1)} = x\sum_{k=0}^{n-1} q_{(n-1)k}(x) = x.$$

$$B_n[x^2](x) = \sum_{j=0}^{n} (\frac{j}{n}C_j^n) x^j (1-x)^{n-j}(\frac{j}{n}) = \frac{x}{n}\sum_{j=1}^{n}(j-1)C_{j-1}^{n-1}x^{j-1}(1-x)^{(n-1)-(j-1)} + \frac{x}{n}$$

$$= \frac{x^2}{n}(n-1)\sum_{j=2}^{n} C_{j-2}^{n-2}x^{j-2}(1-x)^{n-2-(j-2)} + \frac{x}{n} = x^2 + \frac{1}{n}x(1-x).$$

Remark 1. Note that this computes the Bernstein polynomials of $1, x, x^2$. In particular, 1 and x are eigenfunctions of the linear operator B_n in $C[0,1]$, with eigenvalue 1.