FROM THE PARALLELOGRAM LAW TO AN INNER PRODUCT

Let $(V, ||\cdot||)$ be a normed real vector space. Suppose the norm satisfies the *parallelogram law*:

$$||v+w||^2 + ||v-w||^2 = 2||v||^2 + 2||w||^2, \qquad \forall v, w \in V.$$

Define a symmetric form $q : V \times V \to R$ by:

$$q(v,w) = \frac{1}{4}(||v+w||^2 - ||v-w||^2).$$

Theorem. q is an inner product on V.

Proof. Since $q(v,v) = ||v||^2$, clearly q is positive definite; it is clearly also symmetric. Thus we need to verify *additivity*:

$$q(v_1 + v_2, w) = q(v_1, w) + q(v_2, w)$$

and *homogeneity*:

$$q(\lambda v, w) = \lambda q(v,w), \qquad \forall \lambda \in R.$$

Additivity is equivalent to:

$$||v_1+v_2+w||^2 - ||v_1+v_2-w||^2 = ||v_1+w||^2 - ||v_1-w||^2 + ||v_2+w||^2 - ||v_2-w||^2.$$

Exercise 1. Prove this, following the outline below

(a) The parallelogram law applied to the vectors v_1 and $v_2 + w$ gives:

$$||v_1 + v_2 + w||^2 = -||v_1 - w - v_2||^2 + 2||v_1||^2 + 2||v_2 + w||^2.$$

Apply the parallelogram law again to the vectors v_2 and $v_1 - w$ to get a similar identity (with left-hand side $||v_1 + v_2 - w||^2$) and subtract one identity from the other.

(b) Exchange v_1 and v_2 to get a second identity, similar to the one in part (a). Both will have the same left-hand side, namely:

$$||v_1 + v_2 + w||^2 - ||v_1 + v_2 - w||^2.$$

Now add these two identities to get an expression for $4q(v_1 + v_2, w)$, establishing additivity.

Exercise 2. To begin the proof of **homogeneity**, show that

$$q(nv_1, v_2) = nq(v_1, v_2), \qquad n = 1, 2, 3, \ldots$$

and then that:

$$q(\frac{1}{m}v_1, v_2) = \frac{1}{m}q(v_1, v_2) \qquad m = 1, 2, \ldots.$$

Show this implies:

$$q(rv_1, v_2) = rq(v_1, v_2), \qquad \forall r \in \mathbb{Q} \text{ (rationals)}.$$

Exercise 3. Assume the Cauchy-Schwarz inequality holds for q:

$$|q(v, w)| \leq ||v||||w||.$$

Then for $\lambda \in R$ and $r \in \mathbb{Q}$, use:

$$q(\lambda v, w) - \lambda q(v, w) = q(\lambda v, w) - q(rv, w) - (\lambda - r)q(v, w)$$

to show that:

$$|q(\lambda v, w) - \lambda q(v, w)| \leq 2|\lambda - r|||v||||w||.$$

Show how this implies the homogeneity condition for q.

It remains to prove the Cauchy-Schwarz inequality for q.

Exercise 4. Show that, for $r \in \mathbb{Q}$ and $v, w \in V$:

$$r^2||v||^2 + 2rq(v, w) + ||w||^2 \geq 0.$$

Explain why this implies that:

$$4|q(v, w)|^2 - 4||v||^2||w||^2 \leq 0,$$

and hence the Cauchy-Schwarz inequality for q.

Remark. Note that this argument rests on the surprising fact that *additivity implies homogeneity.*