QUADRATIC FORMS/ MINIMIZING PROPERTIES OF EIGENVALUES

In what follows V denotes a finite-dimensional vector space over the real numbers, endowed with an inner product.

1. Quadratic forms.

Definition. A *symmetric bilinear form* on V is a function $B : V \times V \to R$ which is symmetric and bilinear:

$$B(v, w) = B(w, v) \qquad B(\lambda u + v, w) = \lambda B(u, w) + B(v, w).$$

The associated *quadratic form* is the function $Q : V \to R$, $Q(v) = B(v, v)$. Clearly B can be recovered from Q in the usual way:

$$B(v, w) = \frac{1}{4}(Q(v + w) - Q(v - w)).$$

(For this reason, sometimes one refers to B itself as a 'quadratic form'.)

Proposition 1. Given a symmetric bilinear form, there is a unique self-adjoint operator $T \in \mathcal{L}(V)$ such that:

$$B(v, w) = \langle Tv, w\rangle, \qquad \forall v, w \in V.$$

Proof. Let (e_i) be an orthonormal basis of V. Define:

$$Tv = \sum_{i,j} v_i B(e_i, e_j)e_j = \sum_j B(v, e_j)e_j, \qquad \text{where } v_i = \langle v, e_i\rangle.$$

Then:

$$\langle Tv, w\rangle = \sum_j B(v, e_j)\langle e_j, w\rangle = B(v, w) \qquad \forall v, w,$$

and then $T^* = T$ follows from the fact B is symmetric.

Problem 1. Prove that the operator T is uniquely defined, that is: using a different orthonormal basis (f_j) for V in the definition would have led to exactly the same operator $T \in \mathcal{L}(V)$. *Hint:* Use the fact that any two orthonormal bases of V are related by an isometry.

Corollary 1. Let (e_i) be an orthonormal basis of V diagonalizing the self-adjoint operator T associated to a quadratic form Q, with eigenvalues $\lambda_i, i = 1, \ldots, n$ (each eigenvalue listed as many times as the dimension of its eigenspace.) Then:

$$Q(v) = \lambda_1 v_1^2 + \lambda_2 v_2^2 + \ldots + \lambda_n v_n^2, \qquad v_i = \langle v, e_i\rangle, i = 1, \ldots, n.$$

In this case we say Q has been *diagonalized*, or is in *diagonal form.*

Some standard terminology.

Let Q be a quadratic form on a vector space V.

Q is *positive definite* if $Q(v) > 0$. $\forall v \neq 0$; equivalently, all $\lambda_i > 0$. (A positive definite bilinear form on V is the same as an inner product.)

Q is *positive semidefinite* if $Q(v) \geq 0 \ \forall v \in V$ (all $\lambda_i \geq 0$.)

Q is *indefinite* if $Q(v)$ is positive for some v, negative for others (so some $\lambda_i > 0$, some $\lambda_i < 0$.)

Problem 2. Show that if Q is indefinite, there exists $v \in V, v \neq 0$, such that $Q(v) = 0$. (The set of such $v \in V$ is a *cone* in V, that is, invariant under multiplication by real numbers.)

Q is *degenerate* if there exists $v \in V, v \neq 0$ so that $B(v, w) = 0 \ \forall w \in V$. (Equivalently, if 0 is an eigenvalue of T).

Examples 1. It is fairly easy to classify a quadratic form if it is diagonalized.

$Q(v) = x_1^2 + 4x_2^2$ is positive definite in R^2;

$Q(v) = 2x_1^2 - 3x_2^2 + 5x_3^2$ is indefinite in R^3 (and nondegenerate);

A quadratic form q is degenerate exactly when $null(T_q) \neq \{0\}$ (T_q denotes the self-adjoint operator associated to q by an inner product.) And q is positive semidefinite, but not definite, exactly when the eigenvalues of T_f are nonnegative, but 0 is an eigenvalue. Thus if a quadratic form is positive semidefinite, but not positive definite, it is necessarily degenerate.

$Q(v) = 2x_1^2 + 3x_3^2$ in R^3 is positive semidefinite (and degenerate).

$Q(v) = 2x_1^2 - 3x_3^2$ in R^3: indefinite and degenerate.

Note. From now on we will often denote both the bilinear form and its associated quadratic form indiscriminately by q (and refer to them as 'a quadratic form on V'); which one is meant will be clear from context.

Let q be a quadratic form on V, $(e_i)_{i=1}^n$ an orthonormal basis of V diagonalizing q (which by definition means diagonalizing the self-adjoint operator T_q: $T_q e_i = \lambda_i e_i$.) Let V_+, V_- be the subspaces of V spanned by eigenvectors of T_q with positive (resp. negative) eigenvalues, and let $V_0 = Null(T_q)$. We have:

$$V = V_+ \oplus V_- \oplus V_0.$$

This direct sum is q-orthogonal; if v_1, v_2 are in different summands, $q(v_1, v_2) = 0$ (why?) q is positive definite (resp. negative definite) on V_+ (resp. V_-.) And $q(v, u) = 0$ for all $v \in V$, if $u \in V_0$.

Note V_0 is uniquely defined (it is $Null(T_q)$, and T_q is unique, given q.) But V_+ and V_- are not the only maximal subspaces of V on which q is positive (resp. negative) definite. However, the dimensions of such subspaces depend only on q.

Proposition 2. Any subspace $W \subset V$ on which q is positive definite (resp. negative definite) has dimension at most equal to $\dim(V_+)$ (resp $\dim(V_-)$).

Proof of proposition. We claim the sum of subspaces $W_+ + V_- + V_0$ is a direct sum: if $w \in W_+, v \in V_-, u \in V_0$ are such that $w + v + u = 0$, then $w = v = u = 0$. To see this, taking the difference of:

$$0 = q(v, w+v+u) = q(v, v)+q(v, w) \text{ and } 0 = q(w, v+w+u) = q(w, w)+q(w, v)$$

we find $q(w, w) = q(v, v)$, which implies $w = 0$ and $v = 0$ (since otherwise $q(w, w) > 0$ and $q(v, v) < 0$); and therefore also $u = 0$, proving the claim.

Thus:

$$\dim(W)+\dim(V_-)+\dim(V_0) = \dim(W \oplus V_- \oplus V_0) \leq \dim(V) = \dim(V_+)+\dim(V_-)+\dim(V_0),$$

so $\dim(W) \leq \dim(V_+)$, proving the proposition.

Corollary 2. Consider two q-orthogonal direct sum decompositions of V:

$$V = V_+ \oplus V_- \oplus V_0, \qquad V = W_+ \oplus W_- \oplus V_0,$$

with $V_0 = Null(T_q)$, q positive definite on V_+, W_+, negative definite on V_-, W_-. Then:

$$\dim W_+ = \dim V_+, \qquad \dim W_- = \dim V_-.$$

Proof. From the proposition, $\dim(W_+) \leq \dim(V_+)$, and exchanging the two decompositions we get $\dim(V_+) \leq \dim(W_+)$, hence equality; analogous for V_- and W_-.

Thus the largest dimensions of a subspace on which q is positive definite (resp. negative definite) are well defined (depend only on q, not on a diagonalizing orthonormal basis). Their difference $\dim(V_+) - \dim(V_-)$ is called the *signature* of q. Combined with the *rank* of q (which by definition means the rank of T_q, or $\dim(V) - \dim(V_0)$, or $\dim(V_+) + \dim(V_-)$), it determines both $\dim(V_+)$ and $\dim(V_-)$.

Example 2. Consider the quadratic form in R^3 (in standard coordinates):

$$q(v) = x^2 + 10xz + z^2.$$

The matrix of T_q in the standard basis is:

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

Eigenvalues are $6, 0, -4$, with orthonormal basis of eigenvectors:

$$e_1 = \frac{1}{\sqrt{2}}(1, 0, 1) \quad e_2 = (0, 1, 0), \quad e_3 = \frac{1}{\sqrt{2}}(1, 0, -1).$$

Thus in coordinates $u_i = \langle v, e_i \rangle$, the form q is diagonalized:

$$q(v) = 6u_1^2 - 4u_3^2.$$

We see that q is indefinite and degenerate, with rank 2 and signature 0.

Also, $q(v) = 0 \Leftrightarrow 6u_1^2 = 4u_3^2$ (u_2 arbitrary), a cone in R^3.

In addition to $V_+ = span(e_1)$, q is positive-definite on many other one-dimensional subspaces. In fact we only need $\sqrt{3}|u_1| > \sqrt{2}|u_3|$, so $W = \{v; \sqrt{3}u_1 = 2u_3, u_2 = 0\}$ is an example.

Problem 3. Let q be a quadratic form on R^5, with signature 1 and rank 5. Assume the eigenvalues of T_q have absolute value 1.

(i) Write down q in diagonalizing coordinates $(u_1, \ldots, u_5)$;

(ii) Write down an equation (in the coordinates u_i) for the cone $K = \{v \in R^5; q(v) = 0\}$.

(iii) Is there a *subspace* of R^5 on which $q(v) = 0$? Justify.

(iv) Give two examples of three-dimensional subspaces of R^5 on which q is positive-definite.

2. Minimizing properties of eigenvalues. Let $T \in \mathcal{L}(V)$ be self-adjoint. List the eigenvalues of T in increasing order:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

(Each eigenvalue is listed as many times as the dimension of its eigenspace.)

Theorem 1. Let $Q(v) = \langle Tv, v\rangle$ be the associated quadratic form. Then

$$\lambda_1 = \min\{Q(v); ||v|| = 1\} = \min\{\frac{Q(w)}{||w||^2}; w \neq 0.\}$$

Proof. Let (e_i) be an orthonormal basis of V diagonalizing Q ($Te_i = \lambda_i e_i$, where T is the associated self-adjoint operator). For $v \in V$, let $v_i = \langle v, e_i\rangle$. Then:

$$Q(v) = \langle Tv, v\rangle = \sum_i \lambda_i v_i^2 \geq \lambda_1 \sum_i v_i^2 = \lambda_1 ||v||^2 = \lambda_1$$

if $||v|| = 1$. On the other hand, we have $Q(e_1) = \lambda_1$. This concludes the proof.

In the same way we prove:

Proposition 3. For $j = 1, \ldots, n$, let

$$E_j = \oplus\{E(\mu, T); \mu \text{ is an eigenvalue of } T, \lambda_1 \leq \mu \leq \lambda_j\}$$

(the orthogonal direct sum of eigenspaces of T, with eigenvalue μ, $\lambda_1 \leq \mu \leq \lambda_j$.)

Then if $\lambda_{j+1} > \lambda_j$, we have:

$$\lambda_{j+1} = \min\{Q(v); v \in E_j^\perp, ||v|| = 1\} \qquad j = 1, \ldots, n-1.$$

Problem 4. Prove this.

Problem 5. Prove the following: If $T \in \mathcal{L}(V)$ is arbitrary, let $\sigma_1 \geq 0$ be its smallest singular value, and let σ_n be its largest. Then:

$$\sigma_1 = \min\{||Tv||; ||v|| = 1\} \text{ and } \sigma_n = \max\{||Tv||; ||v|| = 1\}.$$