WINDING NUMBER AND APPLICATIONS

We denote by $S^1 \subset R^2$ the unit circle in the plane (or in the complex plane $\mathbb{C}$). Let $c : [0, 2\pi] \to S^1$ be a continuous *closed* curve in the unit circle, $c(0) = c(2\pi)$ (the parametrizing interval chosen is not important.) We want to measure the "net number of turns around the origin" made by c as t varies in $[0, 2\pi]$.

The angle function.

Let $c(t) = (a(t), b(t))$ be a smooth parametrized curve in the plane, $t \in I, 0 \in I$, with values in the unit circle: $a^2 + b^2 \equiv 1$. Let $\varphi_0 \in [0, 2\pi)$ be such that $\cos\varphi_0 = a(0), \sin\varphi_0 = b(0)$. Then the smooth function $\varphi : [0, 2\pi) \to R$:

$$\varphi(t) = \varphi_0 + \int_0^t (ab' - ba')d\tau$$

satisfies $a(t) = \cos\varphi(t), b(t) = \sin\varphi(t), \quad t \in I$.

Proof. Define:

$$F(t) = (a(t) - \cos\varphi(t))^2 + (b(t) - \sin\varphi(t))^2,$$

and show (by differentiation) $F'(t) \equiv 0$.

We refer to $\varphi(t)$ as an 'angle function' for the curve c. (It is unique up to adding a constant.)

Definition. Let $c : [0, 2\pi] \to S^1$ be a closed parametrized curve, with values in the unit circle: $c(0) = c(2\pi), |c(t)| \equiv 1$. Let $\varphi(t)$ be an angle function for c. We define the *winding number* of c by:

$$n(c) = \frac{1}{2\pi}[\varphi(2\pi) - \varphi(0)].$$

Note that since c is closed, $n(c)$ is necessarily an integer (positive or negative).

If $c(t) = (a(t), b(t))$ takes values in $R^2 \setminus \{0\}$, we write $c(t) = r(t)\gamma(t)$ with $r = \sqrt{a^2 + b^2} > 0$ and $\gamma(t)$ taking values in the unit circle, and define $n(c) = n(\gamma)$, using an angle function $\varphi(t)$ for γ.

In fact the winding number can be expressed as the line integral of a closed 1-form on $R^2 \setminus \{0\}$. Let:

$$\omega_0 = \frac{xdy - ydx}{x^2 + y^2}$$

Problem 1. Prove that, for a parametrized curve c taking values in the plane minus the origin:

$$n(c) = \frac{1}{2\pi}\int_c \omega_0.$$

Hint. Use the definition of line integral and the fact $c(t) = (r(t)\cos\varphi(t), r(t)\sin\varphi(t))$ and the corresponding expression for $c'(t)$.

Properties of the winding number.

Recall two closed plane curves c_0, c_1 not meeting the origin (parametrized over $[0, 2\pi]$) are said to be*freely homotopic* if there exists a continuous map $H : [0,1] \times [0, 2\pi] \to R^2 \setminus \{0\}$ so that:

(i) For each fixed $s \in [0,1] : H(s, 0) = H(s, 2\pi)$;

(ii) $F(0,t) = c_0(t), F(1,t) = c_1(t)$. Thus the closed curves $c_s(t) = F(s,t)$ deform c_0 to c_1, without hitting the origin.

(1) If c_0, c_1 are freely homotopic closed curves in $R^2 \setminus \{0\}$, then $n(c_0) = n(c_1)$.

(2) Conversely, if c_0, c_1 are closed plane curves not hitting the origin and $n(c_0) = n(c_1)$, then they are freely homotopic;

(3) $c(t) = (\cos kt, \sin kt); t \in [0, 2\pi]$ satisfies $n(c) = k$, $\forall k \in \mathbb{Z}$;

(4) If c_0, c_1 are closed curves in $R^2 \setminus \{0\}$ and the line segment in R^2 from $c_0(t)$ to $c_1(t)$ does not include the origin (for any t), then $n(c_0) = n(c_1)$. In particular this is true if $|c_0(t) - c_1(t)| < |c_0(t)|$ for each $t \in [0, 2\pi]$.

Proof. (1) follows from the invariance of line integrals of closed 1-forms under free homotopy. (3) is clear (use kt as an angle function). To see (4), use the line segment homotopy from c_0 to c_1:

$$c_s(t) = H(s,t) = (1-s)c_0(t) + sc_1(t), \quad t \in [0, 2\pi], s \in [0, 1.$$

The hypothesis of (4) implies the closed curves c_s do not hit the origin.

To prove (2), assume first c_0 and c_1 take values in the unit circle, and let $\varphi_0, \varphi_1 : [0, 2\pi] \to R$ be choices of angle functions for c_0, c_1 (resp.) Consider:

$$\varphi(s,t) = (1-s)\varphi_0(t) + s\varphi_1(t), \quad H(s,t) = (\cos\varphi(s,t), \sin\varphi(s,t)), \quad s \in [0,1], t \in [0, 2\pi].$$

Note that:

$$\varphi(s, 2\pi) - \varphi(s, 0) = (1-s)2\pi n(c_0) + s2\pi n(c_1) = 2\pi n, \quad \text{since } n(c_0) = n(c_1) := n.$$

Thus the curves $c_s(t) = H(s,t)$ on the unit circle are closed (with same winding number n), and deform c_0 to c_1.

For the general case, just note that a closed curve $c(t)$ on the plane minus the origin and its normalization $\hat{c}(t) = \frac{c(t)}{|c(t)|}$, with values in the unit circle, are freely homotopic via a line segment homotopy avoiding the origin (and by definition $c, \hat{c}$ have the same winding number).

Some applications of the winding number.

1. Topological proof of the Fundamental Theorem of Algebra. Let $p(z)$ be a polynomial of degree $k > 0$ with complex coefficients:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^k, \qquad a_i \in \mathbb{C}, a_k \neq 0,$$

considered as a function $p : \mathbb{C} \to \mathbb{C}$. The *Fundamental Theorem of Algebra* asserts there exists at least one $z_0 \in \mathbb{C}$ so that $p(z_0) = 0$.

Proof. (By contradiction) For $r \geq 0$, let c_r be the image under p of the circle of radius r centered at the origin $0 \in \mathbb{C}$: $c_r(t) = p(re^{it})$, $t \in [0, 2\pi]$. If $p(z) \neq 0 \forall z$, c_r is a curve in $\mathbb{C}^*$ (the complex plane minus the origin), for each $r \geq 0$. We *claim* $n(c_r) = k$ (the degree of p) if r is sufficiently large. This follows easily from (3) above, if $p(z) = a_k z^k$.

In general, we may write $p(z) = a_k z^k + q(z)$, where q is a polynomial of degree $k - 1$, so:

$$|p(z) - a_k z^k| = |a_k z^k||f(z)| \quad f(z) = \frac{q(z)}{a_k z^k} \to 0 \text{ as } z \to \infty.$$

In particular, if $|z| = r > r_0$ (for some $r_0 > 0$), we have: $|p(z) - a_k z^k| < |a_k z^k|$, and using (4) above we see that $n(c_r) = k$ for $r > r_0$, proving the claim.

Now to prove the FTA, just observe that $n(c_r)$ is independent of r (since all the closed curves c_r are freely homotopic to one another, via line-segment homotopies.) But $n(c_0) = 0$, since it is a constant curve. Contradiction.

2. Degree of a map of the circle.

Define $\pi_0 : [0, 2\pi] \to S^1$ by $\pi_0(t) = e^{it}$.

Given $f : S^1 \to S^1$ continuous, let $\varphi : [0, 2\pi] \to R$ be an angle function for the closed curve $c = f \circ \pi_0 : [0, 2\pi] \to S^1$, and define the degree of f by:

$$deg(f) = n(f \circ \pi_0).$$

Note that:

(i) If f is homotopic to a second map $g : S^1 \to S^1$, then the curves $f \circ \pi_0$ and $g \circ \pi_0$ are freely homotopic; hence they have the same winding number, hence $deg(f) = deg(g)$.

(ii) Conversely, if $n(f \circ \pi_0) = n(g \circ \pi_0)$, we know the curves $f \circ \pi_0$ and $g \circ \pi_0$ are homotopic (via $K(s,t), K : [0,1] \times [0,2\pi] \to S^1$) so:

$$H(s, e^{it}) = K(s,t)$$

defines a homotopy from f to g. (Note $K(s,2\pi) = K(s,0)$, so $H(s,x)$, $x \in S^1$, is well-defined by this formula).

The degree of a map indicates if it can be lifted to a map from S^1 to R:

Lemma 1. Let $f : S^1 \to S^1$ be a continuous map. Then $deg(f) = 0$ if and only if It is possible to find a function $\tilde{f} : S^1 \to R$ so that $f(x) = e^{i\tilde{f}(x)}$ on S^1 (if and only if f is homotopic to a constant.)

Proof. First suppose $deg(f) = 0$, and let $\varphi : [0,2\pi] \to R$ be an angle function for the curve $f \circ \pi_0 : [0,2\pi] \to S^1$. Then $\varphi(0) = \varphi(2\pi)$, so we may define $\tilde{f}$ on S^1 by $\tilde{f}(e^{it}) = \varphi(t)$. Then $e^{i\tilde{f}(x)} = e^{i\varphi(t)} = f(x)$ if $x = e^{it}$.

Conversely, if $\tilde{f} : S^1 \to R$ exists with $f(x) = e^{i\tilde{f}(x)}$, then letting $\tilde{f}_s(x) = (1-s)\tilde{f}(x)$ for $s \in [0,1]$, and setting:

$$f_s(x) = e^{i\tilde{f}_s(x)} = e^{i(1-s)\tilde{f}(x)}$$

we define a homotopy from f to a constant map, so $deg(f) = 0$.

Corollary (of lemma 1). If $f : S^1 \to S^1$ (continuous) is homotopic to a constant map, then there exists $z \in S^1$ so that $f(z) = f(-z)$.

Proof. Let $\tilde{f} : S^1 \to R$ be such that $f(z) = e^{i\tilde{f}(z)}$ for all $z \in S^1$. Fix $z_0 \in S^1$; if $\tilde{f}(z_0) = \tilde{f}(-z_0)$, we're done. Otherwise the continuous function $g(z) = \tilde{f}(z) - \tilde{f}(-z)$ from S^1 to R takes values with opposite signs at z_0 and $-z_0$, hence must vanish somewhere on S^1.

3. Brouwer fixed point theorem for the disk in R^2.

Theorem. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ be the closed unit disk. If $f : D \to D$ is continuous, f has a fixed point $z_0 \in D$ ($f(z_0) = z_0$.).

Proof. If there are no fixed points in D, consider the map $g : D \to S^1$ (continuous):

$$g(z) = \frac{f(z) - z}{|f(z) - z|}.$$

It is easy to see g has no fixed points on the unit circle S^1: $g(z) = z$ with $|z| = 1$ implies $f(z) = z(1 + |f(z) - z|)$, so $|f(z)| > 1$, impossible.

But this implies the restriction of g to S^1 is homotopic to the antipodal map $\alpha : S^1 \to S^1$, $\alpha(z) = -z$: just note that $(1 - s)g(z) + s\alpha(z) \neq 0$ if $|z| = 1$ and $s \in [0, 1]$. (Otherwise $(1 - s)|g(z)| = s|\alpha(z)|$, so $s = 1/2$, leading to $g(z) = z$, contradiction.) Thus we may set:

$$H(s, z) = \frac{(1 - s)g(z) + s\alpha(z)}{|(1 - s)g(z) + s\alpha(z)|} \in S^1; \quad |z| = 1, s \in [0, 1].$$

H defines a homotopy from g to α (as continuous maps from S^1 to S^1.)

But on the other hand α is homotopic to the identity map on S^1 (via the deformation $h_s(z) = e^{i\pi s}z$, $s \in [0, 1]$.) In particular the degree of α is 1, and therefore also $deg(g_{|S^1}) = 1$ (winding number of the restriction of g to the unit circle.) But in fact the map $g_{|S^1}$ of the circle is homotopic to a constant: since g is defined on the whole disk, we have the deformation $g_s(z) = g(sz)$, $s \in [0, 1]$, $|z| = 1$, with g_0 the constant map and $g_1 = g_{|S^1}$, implying $deg(g_{|S^1}) = 0$, contradiction. This concludes the proof of Brouwer's fixed point theorem.

Problem 2. Recall a *homeomorphism* $h : \overline{U} \to \overline{V}$ between sets $\overline{U}, \overline{V} \subset R^n$ is a bijective continuous map with continuous inverse. Prove that if $U \subset R^2$ is open and $\overline{U}$ (the closure of U) is homeomorphic to the closed disk D, then any continuous map $f : \overline{U} \to \overline{U}$ has a fixed point in $\overline{U}$.

For the problems below, we use the following definition. Let $S^n \subset R^{n+1}$ be the unit sphere. Two continuous maps $f, g : S^n \to S^n$ are *homotopic* if there exists a continuous map $H : [0, 1] \times S^n \to S^n$ such that $H(0, x) = f(x), H(1, x) = g(x)$, for all $x \in S^n$.

Problem 3. (i) Show that if $f, g : S^n \to S^n$ (continuous) satisfy

$$||f(x) - g(x)|| < 2 \text{ for all } x \in S^n$$

(for the usual norm in R^{n+1}), then f and g are homotopic .

Hint: the hypothesis implies $f(x)$ and $g(x)$ are never antipodal points (prove this), so $(1-s)f(x)+sg(x) \neq 0$, for all $s \in [0,1]$ and all $x \in S^n$ (prove this too.) Now use the norm of this expression as a denominator, to construct a natural homotopy.

(ii) Prove that if $f : S^n \to S^n$ (continuous) has no fixed points on S^n, then f is homotopic to the antipodal map α ($\alpha(x) = -x$). (Look at the hint in part (i).)

(iii) Show that if $f : S^n \to S^n$ (continuous) and $f(x) \neq \alpha(x) \forall x$, then f is homotopic to the identity map.

Problem 4. Prove that if n is odd, the antipodal map $\alpha : S^n \to S^n$ is homotopic to the identity (generalizing the case $n = 1$ used in the proof of Brouwer fixed point, above.)

Hint: Realize S^{2m-1} as the unit sphere in $\mathbb{C}^m$:

$$S^{2m-1} = \{z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m; |z_1|^2 + \ldots + |z_m|^2 = 1\}.$$

4. Borsuk-Ulam Theorem. Let $f : S^2 \to R^2$ be a continuous map. Then there exists $x \in S^2$ so that $f(x) = f(-x)$.

Proof of the Borsuk-Ulam theorem. (By contradiction.)
If $f(x) \neq f(-x) \forall x \in S^2$, consider the map $g : S^2 \to S^1$ (continuous and odd, $g(-x) = -g(x)$):

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Restricting g to the equatorial S^1, we have a map $g_{|S^1} : S^1 \to S^1$, which is homotopic to a constant map (since it extends to the upper hemisphere). By the corollary of lemma 1 (see above), there exists $z \in S^1$ so that $g(z) = g(-z)$. But this contradicts the fact that g is odd on S^2.

5. Index of a planar vector field on the boundary of a disk. (See [dC], p.24–26.)

Let $X : U \subset R^2$ be a smooth vector field on an open set $U \subset R^2$. A *closed orbit* of X is a simple closed curve c (bounding a region homeomorphic to the disk) satisfying: $c'(t) = X(c(t))$. (In particular X does not vanish at points of c.).

Problem 5. Show that the region D bounded by c must contain a zero of X.

Hint. Prove first that $n(X, D)$ (the index of X along $c = \partial D$) equals 1. (Note that the winding number of a simple closed curve enclosing the origin, traversed once counterclockwise, is 1–why?)