THE MÖBIUS STRIP AND ORIENTABILITY

Construction. Consider the open rectangle $R = (0,5) \times (0,1)$ in $\mathbb{R}^2$. For $0 \leq a < b \leq 5$, denote by R_{ab} the open subrectangle $(a,b) \times (0,1) \subset R$. Define in R the equivalence relation:

$$(s,t) \in R_{01} \sim (s+4, 1-t) \in R_{45},$$

(extended as the identity elsewhere on R, so as to be an equivalence relation.) Let $M = R/\sim$ be the set of equivalence classes, and $\pi : R \to M$ the quotient projection (which maps each point to its equivalence class.)

We'll cover M by two paramerizations, which define on M the structure of a two-dimensional manifold. Namely, let:

$$f : R_{03} \to M, \qquad g : R_{25} \to M$$

be the restriction of π to the sets R_{03}, R_{25}, which are open subsets of $\mathbb{R}^2$. It is easy to check that f and g are injective, and that their images cover M (since $R_{03} \cup R_{25} = R$.)

Also, the intersection $W = f(R_{03}) \cap g(R_{25})$ is not empty.

Exercise 1. (i) Explain why $f^{-1}(W) = R_{01} \sqcup R_{23}$ (disjoint union) and $g^{-1}(W) = R_{23} \sqcup R_{45}$.

(ii) Let $F = g^{-1} \circ f : R_{01} \sqcup R_{23} \to R_{23} \sqcup R_{45}$ be the 'transition map'. Compute $F(s,t)$ explicitly, and explain why F is a diffeomorphism.

Thus f, g introduce on M the structure of a 2-dimensional manifold. We'll see that M is not orientable.

Orientability. Recall an n-dimensional manifold M is *orientable* if it admits a smooth structure defined by local parameters $f_\alpha : U_\alpha \to M$ ($U_\alpha \subset R^n$ open), so that on any overlaps $W = f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \neq \emptyset$, the transition diffeomorphism $F = f_\beta^{-1} \circ f_\alpha$ has positive Jacobian determinant throughout $f_\alpha^{-1}(W)$.

Proposition. If M is oriented and $f : N \to M$ is a local diffeomorphism from a second oriented manifold N, and N is connected, then f either preserves or reverses orientation (everywhere on N) . (*Proved in class.*)

In particular, if M is oriented and $f : U \to M$ is a diffeomorphism onto its image $f(U) \subset M$ (where U is a connected open subset of R^n, with its standard orientation), then f is orientation preserving or orientation reversing throughout U; by reversing the orientation in U if necessary, we may assume f is orientation preserving (so U has the orientation induced from that of M via f.)

If now $g : V \to M$ is a second diffeomorphism onto $g(V) \subset M$ (where $V \subset R^n$ is open and connected) we may again induce (via g) the orientation from M on all of V. Now assume the overlap $W = f(U) \cap g(V) \neq \emptyset$. Then the diffeomorphism $F = g^{-1} \circ f$ (from $f^{-1}(W)$ to $g^{-1}(W)$, both open subsets of R^n) must be orientation preserving *throughout* $f^{-1}(W)$. That is, the Jacobian determinant of F must be positive throughout $f^{-1}(W)$ (which may fail to be connected.) This establishes the following result:

Lemma. Let M be a smooth n-dimensional manifold and $f : U \to M$, $g : V \to M$ be diffeomorphisms from *connected* open sets U, V in R^n to their images $f(U), g(V)$ (open subsets of M). Suppose $W = f(U) \cap g(V) \neq \emptyset$. If the sign of the Jacobian determinant of the transition diffeomorphism $F = g^{-1} \circ f$ (from $U_1 = f^{-1}(W) \subset U$ to $V_1 = g^{-1}(W) \subset V$) is not constant throughout U_1, then M *cannot be orientable.*

Remark: under the hypotheses of the Lemma, necessarily U_1 is not connected, although U and V are.

We want to use the Lemma to prove that the Möbius strip is not orientable. It is enough to consider the same maps $f : R_{03} \to M$ and $g : R_{25} \to M$ as in the construction. (Here $U = R_{03}, U_1 = R_{01} \sqcup R_{23}, V = R_{25}, V_1 = R_{23} \sqcup R_{45}$.)

Exercise 2. Show that the sign of the Jacobian determinant of $F = g^{-1} \circ f$ is not constant throughout U_1. This proves that M is not orientable.