PROBLEM SET 2: Solutions

(From Chapter 2 of do Carmo's *Differential Forms and Applications.*)

2. (a) $\int_c \omega = \int_0^1 \omega(c(t))[c'(t)]dt$ and $|\omega(x)| \leq M \forall x \in U$, while $L = \int_0^1 |c'(t)|dt$. Thus:

$$|\int_c \omega| = |\int_0^1 \omega(c(t))[c'(t)]dt| \leq \int_0^1 |\omega(c(t))||c'(t)|dt \leq M \int_0^1 |c'(t)|dt = ML.$$

(b) It is enough to show that $\int_c \omega = 0$ for any closed curve c in $R^2 \setminus \{0\}$ ($c : [0,1] \to R^2 \setminus \{0\}, c(0) = c(1)$). Note that for $s \in [0,1)$ the curve:

$$c_s(t) = (1-s)c(t), \quad t \in [0,1]$$

is a closed curve in $R^2 \setminus \{0\}$, freely homotopic to c in $R^2 \setminus \{0\}$. Its length is given by:

$$L[c_s] = \int_0^1 |c_s'(t)|dt = (1-s)\int_0^1 |c'(t)|dt = (1-s)L[c],$$

Thus, by part (a): $|\int_{c_s} \omega| \leq (1-s)ML[c]$. But this line integral has the same value as the one over c, hence:

$$|\int_c \omega| \leq (1-s)L[c],$$

which can be made as small as desired by taking s sufficiently close to 1. Hence its value is zero.

(c) The hypothesis in (c) implies $|\omega| \leq \frac{M(r)}{r}$ on the circle S_r of radius $r > 0$, center 0 (where $M(r) \to 0$ as $r \to 0_+$). Any closed curve c in $R^2 \setminus \{0\}$ with winding number k is freely homotopic to S_r^k, the circle S_r traversed k times (in a direction depending on the sign of k.) Note that the length of S_r^k (as a parametrized curve) is $L[S_r^k] = 2|k|\pi r$. Thus, from part (a):

$$|\int_c \omega| = |\int_{S_r^k} \omega| \leq \frac{M(r)}{r} L[S_r^k] = 2|k|\pi M(r),$$

which can be made as small as we want, taking $r > 0$ small enough (since $M(r) \to 0$). Thus $\int_c \omega = 0$.

4. It is enough to show that ω is locally exact (exact in a neighborhood of each point), since locally exact forms are closed. Let $p \in U$, and let

$V = D_{r_0} \subset U$ be the open disk with center p, bounded by the circle with center p, radius r_0 (this is contained in U if r_0 is small enough.) Let c be a closed curve contained in V. Consider the closed curves in V: $c_s(t) = sp + (1-s)c(t)$, for $s \in [0,1]$. The line integral of ω along c_s satisfies:

$$\int_{c_s} \omega = \int_0^1 \omega(c_s(t))c_s'(t)dt = (1-s)\int_0^1 \omega(c_s(t))c'(t)dt,$$

and hence defines a continuous function of $s \in [0,1]$, bounded in absolute value by $(1-s)ML[c]$, where M is the maximum of $|\omega|$ over the closed disk $\bar{D}_{r_0}$ (see problem 2(a)). A continuous function on an interval of $\mathbb{R}$ taking only rational values must be constant, and since its value is zero at $s = 1$ (constant curve), it must be zero for all s. In particular considering $s = 0$ we see that $\int_c \omega = 0$. Since c is an arbitrary closed curve in V, we see that ω is exact in V, hence locally exact in U.

5. Let f, g be potential functions for ω in U, V (resp.): $df = \omega$ in U, $dg = \omega$ in V. Then $d(f-g) \equiv 0$ in $U \cap V$, and since $U \cap V$ is connected it follows that $f = g + C$ in $U \cap V$, for some constant $C \in \mathbb{R}$. Then letting $h = f$ in U, $h = g + C$ in V, we see that h is well-defined in all of $U \cup V$, and satisfies $dh = \omega$. So ω is exact in $U \cup V$.

8. $F : U \to R^2$ is a vector field in $U \subset R^2$, satisfying $F(-q) = -F(q)$ in $D \subset U$, a closed disk with center $0 \in U$. Assume $F \neq 0$ on $c = \partial D$. *Claim:* The index $n(F; D)$ is an odd integer.

To see this, parametrize ∂D (a circle of radius r centered at $0 \in R^2$) by $c : [0, 2\pi] \to U, c(t) = re^{it}$, and write $F(c(t)) = |F(c(t))|e^{i\phi(t)}$, where $\phi : [0, 2\pi] \to \mathbb{R}$ is an angle function for the closed curve with values on the unit circle $F(c(t))/|F(c(t))|$. By definition, $n(F; D) = (\phi(2\pi) - \phi(0))/(2\pi)$. The condition $F(-q) = -F(q)$ for $q \in c$ implies:

$$\phi(t + \pi) = \phi(t) + k\pi, \quad \forall t \in [0, 2\pi],$$

where k is an odd integer. In particular $\phi(2\pi) = \phi(\pi) + k\pi$ and $\phi(\pi) = \phi(0) + k\pi$. Thus $\phi(2\pi) - \phi(0) = 2k\pi$, so $n(F; D) = k$, an odd integer (in particular not 0). That F has a zero in D then follows from Prop. 4 in the text.