

## FINITE-DIMENSIONAL LINEAR SPACES

**Theorem.** Any two norms on a finite-dimensional linear space  $E$  are equivalent.

To see this, let  $\{e_1, \dots, e_N\}$  be a basis of  $E$ , and define a norm  $\|\cdot\|_1$  on  $E$  via:

$$\|v\|_1 = |x_1| + \dots + |x_N|, \quad v = x_1e_1 + \dots + x_Ne_N.$$

*Claim.* Any norm  $p$  on  $E$  is equivalent to  $\|\cdot\|_1$ . To see this, first note:

$$p(v) \leq |x_1|p(v_1) + \dots + |x_N|p(v_N) \leq C\|v\|_1, \quad \text{where } C = \max_{1 \leq i \leq N} p(e_i) > 0.$$

Note that the opposite inequality  $p(v) \geq A\|v\|_1$  is equivalent to:  $p(w) \geq A$  if  $\|w\|_1 = 1$ . Proceeding by contradiction: if this fails for all  $A > 0$ , then we may find a sequence  $v_n \in E$  so that  $\|v_n\|_1 = 1$  for all  $n \geq 1$ , but  $p(v_n) \rightarrow 0$ .

For simplicity of notation, assume  $\dim E = N = 2$ :

$$v_n = x_n^1e_1 + x_n^2e_2, \quad |x_n^1| + |x_n^2| = 1 \text{ for all } n.$$

Then either  $|x_{n_j}^1| \geq 1/3$  for a subsequence, or  $|x_{n_j}^2| \geq 1/3$  for a subsequence; assume the former.

Since  $x_{n_j}^1, x_{n_j}^2$  are bounded sequences of real numbers, by the Bolzano-Weierstrass theorem for  $R$  (or for  $R^2$ ), we may find a further subsequence of  $x_{n_j}$  (still denoted  $x_{n_j}$ ) so that  $x_{n_j}^1 \rightarrow x_0^1, x_{n_j}^2 \rightarrow x_0^2$ . Note  $x_0^1 \neq 0$  (since  $|x_{n_j}^1| \geq 1/3$ ), so setting  $v_0 = x_0^1e_1 + x_0^2e_2$  we have  $v_0 \neq 0$ .

But  $p(v_{n_j} - v_0) \leq |x_{n_j}^1 - x_0^1|p(e_1) + |x_{n_j}^2 - x_0^2|p(e_2) \rightarrow 0$ . Thus we have  $v_{n_j} \rightarrow v_0$  with respect to the norm  $p$ , contradicting  $p(v_n) \rightarrow 0$ , since  $v_0 \neq 0$ .

Thus we have, for two constants  $A > 0, C > 0$ :

$$A\|v\|_1 \leq p(v) \leq C\|v\|_1, \quad \forall v \in E,$$

and the theorem is proved.

### Recovering a norm from its unit ball.

Let  $p : E \rightarrow R_+$  be a norm on a finite-dimensional linear space  $E$ ; consider its closed unit ball:

$$B = \{v \in E; p(v) \leq 1\}.$$

it is easy to see  $B$  is convex, closed, bounded (with respect to  $p$ , and hence with respect to any norm on  $E$ , by equivalence of norms.) It is also ‘balanced’:

$$0 \in B, \text{ and } v \in B \Rightarrow \lambda v \in B, \text{ if } \lambda \in R, |\lambda| \leq 1.$$

For any  $v \in E$ , consider the set  $I(v) \subset R$ :

$$I(v) = \{t > 0; \frac{1}{t}v \in B\}.$$

Clearly  $I(0) = (0, \infty)$ , and otherwise  $I(v) = [n(v), +\infty)$  for some  $n(v) > 0$ .

*Exercise 1.*(i) Show that for any bounded, closed, convex, balanced set  $B$  in  $R^N$ , and any  $v \in B$ , the set  $I(v)$  is not empty, and is a closed interval of the form  $[n(v), +\infty)$  if  $v \neq 0$ , while  $I(0) = (0, \infty)$ .

(ii) Show that if such a set  $B$  is the closed unit ball of a norm  $p$  in  $R^N$ , then  $n(v) = p(v)$ .

In general, given a closed, bounded, convex, balanced set  $B \subset E$ , define a function:

$$p : E \rightarrow R_+; \quad p(v) = \inf I(v),$$

with  $I(v)$  defined as above.

**Claim.** The function  $p$  defines a norm on  $E$ .

*Proof.* (i) Since  $0 \in B$ ,  $p(0) = 0$ . If  $p(x) = 0$ , then  $I(x) = (0, +\infty)$ , so  $x = 0$ ; otherwise the ray  $\{\lambda x; \lambda > 0\}$  would be an unbounded subset of  $R^n$  contained in  $B$ , contradicting the fact  $B$  is bounded.

(ii) For  $\lambda > 0$ , we have (setting  $t = \lambda s$ ):

$$p(\lambda x) = \inf\{t > 0; \frac{\lambda x}{t} \in B\} = \inf\{\lambda s; s > 0, \frac{x}{s} \in B\} = \lambda \inf\{s > 0; \frac{x}{s} \in B\} = \lambda p(x).$$

And similarly, now setting  $s = -t$ :

$$p(-x) = \inf\{t > 0; \frac{(-x)}{t} \in B\} = \inf\{-s > 0; \frac{x}{s} \in B\} = \inf\{s > 0; \frac{x}{s} \in B\} = p(x).$$

Thus  $p(\lambda x) = |\lambda|p(x)$ , for all  $\lambda \in R$ .

(iii) Given  $x, y \in E$  we have, for any  $\epsilon > 0$ :

$$\frac{x}{p(x) + \epsilon} \in B, \quad \frac{y}{p(y) + \epsilon} \in B.$$

But clearly:

$$\frac{x+y}{p(x) + p(y) + 2\epsilon} = \lambda_1 \frac{x}{p(x) + \epsilon} + \lambda_2 \frac{y}{p(y) + \epsilon},$$

where

$$\lambda_1 = \frac{p(x) + \epsilon}{p(x) + p(y) + 2\epsilon}, \lambda_2 = \frac{p(y) + \epsilon}{p(x) + p(y) + 2\epsilon}.$$

Since  $\lambda_1 + \lambda_2 = 1$ , convexity of  $B$  implies  $\frac{x+y}{p(x)+p(y)+2\epsilon} \in B$ . Since  $\epsilon > 0$  is arbitrary, this shows:

$$p(x+y) \leq p(x) + p(y),$$

proving the claim.

*Remark.* Note that

$$p(x) \leq 1 \Leftrightarrow I(x) \supset [1, +\infty) \Leftrightarrow x \in B,$$

so  $B$  is the closed unit ball of the norm  $p$ .