Nowhere Differentiable Functions Are Generic in the Space of Continuous Functions

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1 Introduction

In mathematics, one of the most fundamental concepts is that of a set. Most rigorous fields of mathematics understand the objects they are studying as members of some set, and when studying ideal properties of these objects, it is often useful to consider the subset of objects which hold this property and analyze this. Topology is a very useful tool for understanding how sets are structured, and as a result it is very useful for understanding the behavior of such sets of objects. In this paper, we will discuss a property of continuous functions; in fact, we will show that nowhere differentiable functions are **generic** in the space of continuous functions. Roughly speaking, this means that if you chose a continuous function at random, you would be practically guaranteed to choose one which is *nowhere differentiable*, which is a quite surprising result.

2 Necessary Concepts

In order to talk about a concept being generic, we first need to give a more rigorous definition of what the term means. To do this, we need a few other definitions. We will assume knowledge of definitions such as open sets and dense sets.

Definition: Let X be a topological space. A subset G of X is called *thick* if we can find a countable collection G_n of open, dense subsets of X such that

$$G = \cap G_n$$

We will say that a property is generic in a topological space X if the set of objects which have the property is thick in X. But first, let us justify this definition. Density seems like a clear choice for the definition, as we want this property to be spread across the set in a dense way. Furthermore, openness helps verify that the instances of this property are not isolated. That is, we have some wiggle room around any point with the desired property where we won't lose that property. So open-dense sets are a natural choice for the idea of a generic property. To justify the inclusion of sets which may not themselves be open-dense, but are the union of countably many open-dense sets, we introduce a theorem.

Baire's Theorem: Every thick subset of a complete metric space M is dense.

We will use these concepts to prove that "Nowhere Differentiable" is a generic property of the space of continuous functions. Below we list some other theorems and concepts we will use in the proof.

Lemma: Let X be a complete metric space. For A, B subsets of X, if
A $\subseteq \overline{B}$ and A is dense in X, then B is dense in X.

Weierstrass Approximation Theorem: The set of polynomials P is dense in C^0 (the space of continuous functions on R).

Before we can begin our proof, we must take a moment to establish one more tool we will need.

3 Sawtooth Functions

Define a **sawtooth function** $\sigma_0 : R \to \text{R}$ by

$$\sigma_0(x) = \begin{cases} x - 2n & 2n \leq x \leq 2n+1 \\ 2n+2-x & 2n+1 \leq x \leq 2n \end{cases}$$

Note that $\sigma_0(x+2m) = \sigma_0(x)$ for all integers m, so σ_0 is periodic with period 2. Also, note that $0 \leq \sigma_0(x) \leq 1$ for all x $\in R$.

Observe that given a desired "size" ϵ and period π, we can supply a sawtooth function which has that size and period. Namely...

$$\sigma_{(\epsilon,\pi)}(x) = \epsilon\sigma_0(\frac{2x}{\pi})$$

Note now the period of $\sigma_{(\epsilon,\pi)}$ is π, and $0 \leq \sigma_{(\epsilon,\pi)} \leq \epsilon$ for all x $\in$ R. Being able to construct these sawtooth function will be helpful in our proof, but first, we can use it to construct our first example of a function which is nowhere differentiable.

For each k $\in \{0, 1, 2...\}$ define

$$\sigma_k(x) = (\frac{3}{4})^k\sigma_0(4^k x)$$

Note then that the "size" of σ_k is $(\frac{3}{4})^k$ and the period is $\frac{2}{4^k}$. Since $0 \leq \sigma_k(x) \leq (\frac{3}{4})^k$ for all x $\in$ R, we have that $\sum_{k=1}^{\infty}\sigma_k(x)$ converges by the Weierstrass M-Test, and it does so uniformly to some limit f. So we may define a continuous function by

$$f(x) = \sum_{k=1}^{\infty}\sigma_k(x)$$

Claim: f is nowhere differentiable.

Fix x ∈ R and define $\delta_n = \frac{1}{2*4^n}$ for all n. Note that $\delta_n \to 0$ as $n \to \infty$, but the difference quotient

$$\frac{\Delta f}{\Delta x} = \frac{f(x \pm \delta_n) - f(x)}{\delta_n}$$

does not converge to a limit. We will argue this below.

Note, we can rewrite our difference quotient as

$$\frac{\Delta f}{\Delta x} = \frac{\sum_{k=1}^{\infty} \sigma_k(x \pm \delta_n) - \sigma_k(x)}{\delta_n}$$

For $k > n$, δ_n is a multiple of the period of σ_k, so all of the summands for $k > n$ vanish. So we need only consider the finite sum up to n. Hence

$$\frac{\Delta f}{\Delta x} = \frac{\sum_{k=1}^{n} \sigma_k(x \pm \delta_n) - \sigma_k(x)}{\delta_n} \tag{1}$$

Furthermore, σ_n is monotone on intervals of length 4^{-n}, and $[x-\delta_n, x+\delta_n]$ is of length 4^-n, so we may conclude that σ_n is monotone on either $[x-\delta_n, x] or [x, x+\delta_n]$ with slope equal to $\pm 3^n$. So we may refine (1) to

$$\frac{\Delta f}{\Delta x} = 3^n + \frac{\sum_{k=1}^{n-1} \sigma_k(x \pm \delta_n) - \sigma_k(x)}{\delta_n} \tag{2}$$

Finally, we can estimate the terms where $k < n$ as less than or equal to the slopes of the monotone portions of σ_k, which are all equal to 3^k. So the absolute value of each of these different quotients is less than or equal to 3^k, but for our purposes, we can use only one side of the inequality to obtain that when $k < n$

$$\frac{\sigma_k(x + \delta_n) - \sigma_k(x)}{\delta_n} \geq 3^k \tag{3}$$

Combining (2) and (3), we obtain the following

$$\frac{\Delta f}{\Delta x} \geq 3^n - (3^{n-1} + 3^{n-2} + ... + 1) = 3^n - \frac{3^{n-1}}{3-1} = \frac{1}{2}(3^n + 1) \tag{4}$$

Of course, the right end of this inequality approaches zero, and so we conclude that $\frac{\Delta f}{\Delta x}$ does the same. Hence the derivative does not exist for any point x ∈ R.

4 Statement and Proof of Theorem

Theorem: A generic f ∈$C^0 = C^0([a, b], R)$ is differentiable at no point of [a,b]. In fact, it is neither left nor right-differentiable at any point in [a,b].

Clearly, proving the latter statement will imply the first. To prove the second statement, define the following two families of sets. For each n ∈ {1,2,3...}, define two sets

$$R_n = \{f \in C^0 : \forall x \in [a, b - 1/n] \exists h > 0 such that |\frac{\Delta f}{h}| > n\}$$

$$L_n = \{f \in C^0 : \forall x \in [a + 1/n, b] \exists h < 0 such that |\frac{\Delta f}{h}| > n\}$$

Claim: $\bigcap_{n=1}^{\infty} R_n \cap L_n$ is a set consisting only of nowhere differentiable functions.

Proof: Let f $\in \bigcap_{n=1}^{\infty} R_n \cap L_n$. Then for each x $\in$ [a,b] there are sequences h_n^+ and h_n^- such that $h_n^- < 0 < h_n^+$ and

$$|\frac{f(x + h_n^-) - f(x)}{h_n^-}| > n \qquad |\frac{f(x + h_n^+) - f(x)}{h_n^+}| > n$$

f is continuous on a compact interval [a,b], so f is bounded. Hence the numerators of these fractions are at most $2||f||$. We can use this estimate and rearrange both of these difference quotients to achieve that

$$|h_n^{\pm}| \leq \frac{2||f||}{n} \quad \forall n \in \{1, 2, 3...\}$$

Then we may conclude that both $h_n^+ \to 0$and $h_n^- \to 0$ as $n \to \infty$. But clearly both of the difference quotients defining the derivative will diverge to infinity. Hence f is not differentiable at x, and since x was an arbitrary point of [a,b], f is not differentiable at any point of [a,b].

All that is left to prove then is that $\bigcap_{n=1}^{\infty} R_n \cap L_n$ is thick in C^0. Equivalently, that both R_n and L_n are open and dense in C^0 for all n $\in \{1, 2, 3...\}$.

To check the denseness, we can use the lemma listed in the necessary concepts section, and so it suffices to show that the closures of R_n and L_n contain the set P of polynomials (which is dense in C^0 by the Weierstrass Approximation Theorem).

Fix n, fix a p $\in$ P and let $\epsilon > 0$. Consider a sawtooth function σ as defined above which has a period smaller than 1/n and size smaller than ϵ, and also satisfying that

$$\min_x\{|slope_x|\} > n + \max_x\{slope_x(P)\}$$

(We can do this by starting with, $\sigma = \frac{\epsilon}{2}\sigma_0(4nx)$), and compressing the function into a smaller period until the slope is larger than the max slope of p, for example).

Now, the slopes of σ are far greater than those of p, so the slopes of $f = p+\sigma$ alternate in sign, and their period is less than 1/2n. At any x $\in$ [a,b], we have that f has a rightward slope either larger than n or smaller than -n. Thus we conclude that f $\in R_n$. Similarly, f $\in L_n$, and so we have found that for any $\epsilon > 0, \exists f \in R_n \cap L_n with ||f - p|| < \epsilon$. So $p \in \overline{R_n} and p \in \overline{L_n}$. This is true for all p $\in$ P, so we find that $P \subseteq \overline{R_n}$ and $P \subseteq \overline{L_n}$

Now we only need that R_n and L_n are open. Let $f \in R_n$. For each x in [a,b], there is an $h = h(x) > 0$ such that

$$|\frac{f(x + h) - f(x)}{h}| > n.$$

Since f is continuous, there is a neighborhood T_x of x in [a,b] and a constant $v = v(x) > 0$ such that the same h will yield

$$|\frac{f(t+h) - f(t)}{h}| > n + v$$

for all $t \in T_x$. Since [a,b-1/n] is compact, finitely many of the T_x cover it, say $T_{x_1}, ..., T_{x_m}$. By continuity of f, we can state that for all $t \in \overline{T}_{x_i}$ we have

$$|\frac{f(t+h_i) - f(t)}{h}| \geq n + v_i,$$

where $h_i = h(x_i)$ and $v_i = v(x_i)$. Now, if we replace f by a function g with d(f,g) small enough, these inequalities hold in a slightly weaker way, and we can still conclude that

$$|\frac{g(t+h_i) - g(t)}{h_i}| > n$$

Hence g $\in R_n$ and R_n is open in C^0. By a very similar reasoning, we have that L_n is open in C^0. Therefore we have that the set of nowhere differentiable functions can be written as a countable intersection of open-dense sets, therefore nowhere differentiable functions are generic in the space of continuous functions.