Topological Characterization of the Segment and Circle

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1 Introduction

In topology, many central problems are determining necessary and sufficient conditions under which two spaces are homeomorphic. The real line $\mathbb{R}$ and its closed subset $[0,1]$ are two of the most fundamental topological spaces, and hence it makes sense to consider the characteristics of spaces which are homeomorphic to them. In this paper, we define characteristics of spaces which have these properties.

2 Definitions and Notation

Definition 1 (Partition)**.** Let X be a topological space. The pair of subsets $P, Q \subsetneq X$ are said to be a *partition* of the subset $A \subseteq X$ if

(a) $P \cap A$ and $Q \cap A$ are open in A,

(b) $P \cap Q = \emptyset$,

(c) $P \cup Q \supseteq A$,

(d) $P \cap A \neq \emptyset$, and

(e) $Q \cap A \neq \emptyset$.

We write $A = P \mid Q$ if the pair (P, Q) form a partition of A.

Definition 2 (Connectedness)**.** Let X be a topological space. A subset $A \subseteq X$ is said to be *disconnected* if there exists a partition of A. Otherwise, A is said to be *connected*.

Definition 3 (Component)**.** Let X be a topological space and $A \subseteq X$ be a subset. A *component* of A is a subset $B \subseteq A$ such that B is maximally connected, that is, if $\exists C$ such that $B \subseteq C$ and C is connected, then $B = C$.

Definition 4 (Cut Point)**.** A *cut point* of a connected set A is a point $x \in A$ such that $A \setminus \{x\}$ is disconnected.

Definition 5 (Exceptional Point)**.** Any point which is not a cut point of a set is an *exceptional point.*

Definition 6 (Continuum)**.** A compact, connected set with at least two points is called a *continuum.*

Definition 7 (Closed Arc)**.** A set X is called a *closed arc* if X is homeomorphic to the interval $[0, 1]$.

Definition 8 (Open Arc)**.** A set X is called a *open arc* if X is homeomorphic to the interval $(0, 1)$.

Definition 9 (Simple Closed Curve)**.** A set X is called a *simple closed curve* or *Jordan curve* if X is homeomorphic to the unit circle.

Definition 10 (Order)**.** Let X be a topological space. An *order* on X is a relation $\prec$ which has the following:

(a) $\nexists x \in X$ for which $x \prec x$, and

(b) $\forall x, y, z \in X$ if $x \prec y$ and $y \prec z$, then $x \prec z$.

We say x *precedes* y if $x \prec y$, z is *between* x and y if $x \prec z \prec y$ or $y \prec z \prec x$, and y is the *successor* of x if $x \prec y$.

Definition 11 (Total Order)**.** Let X be a topological space and $\prec$ be an order on X. We call $\prec$ a *total ordering* if the additional condition

(c) If $x \neq y$, then either $x \prec y$ or $y \prec x$.

is satisfied. If $\prec$ is not a total order, then it is called a *partial order.*

Definition 12 (Section)**.** Let X be a separable topological space with the total ordering $\prec$ and E_0 a countable dense subset. A subset $\Lambda \subseteq E_0$ is called a *section* if

(a) it has no last point, and

(b) if $x \in \Lambda$, for all $y \in E_0$ such that $y \prec x$, $y \in \Lambda$.

Example 1. Let $X = [0, 1]$ and $E_0 = [0, 1] \cap \mathbb{Q}$. If $a \in [0, 1]$, the set $\Lambda = [0, a) \cap \mathbb{Q}$ is a section.

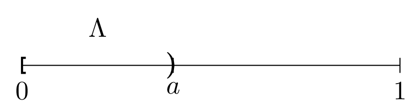

Figure 1: The section $[0, a) \cap \mathbb{Q}$ of $[0, 1]$.

3 Characterization of the Segment and Line

Theorem 1. *If A is connected, but $A \setminus \{x\}$ has the partition $U \mid V$, then $\overline{U} = U \cup \{x\}$ and $\overline{V} = V \cup \{x\}$.*

Proof. Since U is closed in $A \setminus \{x\}$, we have

$$U = \overline{U} \cap (A \setminus \{x\}) = \overline{U} \setminus \{x\},$$

and therefore $\overline{U} \subseteq U \cup \{x\}$. Similarly $\overline{V} \subseteq V \cup \{x\}$. If $\overline{U} = U$, U and $\overline{V} \cup \{x\}$ are two nonempty closed sets whose union is A and whose common part is

$$U \cap \left(\overline{V} \cup \{x\}\right) = U \cap \overline{V} \subseteq U \cap (V \cup \{x\}) = \emptyset,$$

contradicting the assumption that A is connected. □

Theorem 2. *If X is a continuum with two or fewer non-cut points, then X is a closed arc.*

Remark 1. The compactness condition given in continuum is used not only to distinguish $[0,1]$ from $(0,1)$ and $\mathbb{R}$, but also to exclude certain entirely different spaces which satisfy the cut point condition of the previous theorem.

For example, consider the set $A \subset \mathbb{R}^2$ consisting of the curve $\sin\left(\frac{1}{x}\right)$ for $0 < x \leq 1$ and the interval $[-1,0] \times \{0\}$. This is a connected set and has two non-cut points, but it is not locally connected, so it is not a closed arc.

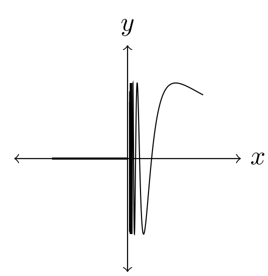

Figure 2: The connected set A with two non-cut points.

Proof. Let X be a continuum with two or fewer non-cut points and $A \subsetneq X$ be the set of exceptional points of X. By assumption, A has 0, 1, or 2 elements. Let $x_0 \in X \setminus A$ and $P \mid Q$ a partition of $X \setminus \{x_0\}$. Since x_0 is not an exceptional point, it is a cut point. Then this means that this partition $P \mid Q$ exists.

By the previous theorem, we have $\overline{P} = P \cup \{x_0\}$ and $\overline{Q} = Q \cup \{x_0\}$. Further note that by definition of partition, P and Q are open.

Claim 1. *$\overline{P}$ and $\overline{Q}$ are connected.*

Proof. Suppose that $\overline{Q}$ is disconnected. Then $\overline{Q} = H_1 \mid H_2$ is a partition. Assume that $x_0 \in H_1$. Then we have

$$H_2 \cap \overline{P} = H_2 \cap (P \cup \{x_0\}) = H_2 \cap P = \emptyset$$

since $P \cap H_2 \subseteq P \cap Q = \emptyset$. Then we have $X = H_2 \mid (H_1 \cup \overline{P})$ since

$$\begin{aligned} H_2 \cap (H_1 \cup \overline{P}) &= (H_2 \cap H_1) \cup (H_2 \cap \overline{P}) = \emptyset \\ H_2 \cup (H_1 \cup \overline{P}) &= \overline{Q} \cup \overline{P} = X \\ H_2 &\neq \emptyset \\ H_1 \cup \overline{P} &\neq \emptyset \end{aligned}$$

and H_2 and $H_1 \cup \overline{P}$ are open. This is a contradiction since we assumed that X is connected. The proof that $\overline{P}$ is connected is similar. □

Lemma 1. *If $A \subsetneq X$ is a connected set and X has the partition $X = P \mid Q$, then $A \subseteq P$ or $A \subseteq Q$.*

Proof. Let $A \subsetneq X$ be a connected set and X have the partition $X = P \mid Q$. Assume that A is not strictly contained in either P or Q, that is, A intersects both P and Q. Then we have

$$\begin{aligned} (P \cap A) \cup (Q \cap A) &= A \cap (P \cup Q) = A \cap X = A \\ (P \cap A) \cap (Q \cap A) &= A \cap (P \cap Q) = \emptyset \\ P \cap A &\neq \emptyset \\ Q \cap A &\neq \emptyset \end{aligned}$$

where $P \cap A$ and $Q \cap A$ are open in A, giving us the partition

$$A = (P \cap A) \mid (Q \cap A),$$

a contradiction since A is assumed to be connected. □

Claim 2. *If $y \in P$ and $P_1 \mid Q_1$ is any partition of $X \setminus \{y\}$, then either $P_1 \subseteq P$ or $Q_1 \subseteq P$.*

Proof. The connected set $\overline{Q}$ is contained in $X \setminus \{y\}$ since $y \in P$. Note that this implies that $\overline{Q}$ is contained in either P_1 or Q_1 by Lemma 1. Then the other part of the partition is contained in $P = X \setminus \overline{Q}$. □

Claim 3. *P contains at least one exceptional point.*

Proof. Suppose that it does not. Then $\overline{P} = P \cup \{x_0\}$ also contains no exceptional point since x_0 is not an exceptional point. Since X is a compact metric space, it is separable. So now we can find $\{x_1, x_2, \ldots\}$ a countable dense set of points in P. By the previous claim, if we have the partition $X \setminus \{x_1\} = P_1 \mid Q_1$, either

$P_1 \subseteq P$ or $Q_1 \subseteq P$. Label the parts such that $P_1 \subseteq P$. We now assume that for $k > 1$, the set P_k has been defined such that

$$X \setminus \{x_{n_k}\} = P_k \mid Q_k$$

and $P_k \subseteq P$. By definition of partition, $P_k \neq \emptyset$, and hence contains at least one point of our countable dense set. By the well-ordering principle of $\mathbb{N}$, we choose n_{k+1} to be the smallest such natural number with $x_{n_{k+1}} \in P_k$. Since $P_k \subseteq P$, $x_{n_{k+1}}$ is not an exceptional point by assumption. Therefore it is a cut point, so $X \setminus \{x_{n_{k+1}}\}$ is disconnected. We have the partition $X \setminus \{x_{n_{k+1}}\} = P_{k+1} \mid Q_{k+1}$ and label these parts such that $P_{k+1} \subseteq P_k$ and hence $P_{k+1} \subseteq P$. Note that since $x_{n_{k+1}} \in P_k$ but $x_{n_{k+1}} \notin P_{k+1}$, we have $P_{k+1} \subsetneq P_k$.

Now we have inductively defined the decreasing sequence $\{P, P_1, P_2, \ldots\}$. Note that since the closure of each of these sets is just the inclusion of the corresponding point that renders X disconnected, we have that $\{\overline{P}, \overline{P_1}, \overline{P_2}, \ldots\}$ is a decreasing sequence of compact sets. By Cantor's intersection theorem, we have that

$$P_\infty = P \cap \bigcap_{i=1}^{\infty} \overline{P}_i \neq \emptyset.$$

Furthermore, since $\overline{P}_{k+1} = P_{k+1} \cup \{x_{n_{k+1}}\} \subseteq P_k$, we also have that

$$P_\infty = P \cap \bigcap_{i=1}^{\infty} P_i.$$

Now let $z \in P_\infty$. Since $z \in P$, z is not an exceptional point. Thus we have the partition $X \setminus \{z\} = H_1 \mid H_2$. By Claim 2, we have that either every P_n contains H_1 or every P_n contains H_2. Label the one that is contained in every P_n by H_1. Then we have that since $H_1 \subset P_n$ for every n, $H_1 \subset P_\infty$. Since H_1 is open and nonempty, it contains at least one point of our countable dense set, say x_i. Let n_{k+1} be the first integer of the $\{n_2, n_3, \ldots\}$ to exceed i. But $x_i \in P_k$, and so n_{k+1} is not the least integer m such that $x_m \in P_k$, which is a contradiction to how we defined the points x_{n_k} and sets P_k. Therefore P must contain an exceptional point.

The proof that Q must contain an exceptional point is similar. □

Note now that since both P and Q have at least one exceptional point, and P and Q are disjoint, the set X must have at least 2 exceptional points. We denote these a and b.

Claim 4. *If x is not an exceptional point, $X \setminus \{x\}$ has two components, each containing one of the two exceptional points.*

Proof. Let x be a cut point of X. We have the partition $X \setminus \{x\} = P \mid Q$. Assume that $a \in P$ and $b \in Q$. We want to show that P and Q are connected. Suppose that we have the partition $P = H_1 \mid H_2$ and that $a \in H_1$. Since H_2 is clopen in P and P is clopen in $X \setminus \{x\}$, we have the partition

$$X \setminus \{x\} = H_2 \mid ((X \setminus \{x\}) \setminus H_2).$$

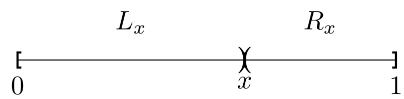

Figure 3: The sections of X that give an ordering.

But this is impossible since H_2 contains neither a nor b, contradicting Claim 3.

The proof that Q is connected is similar. □

The claims proved so far allows us to set up an order on X.

For each $x \in X$ we define L_x to be the empty set if $x = a$ and the component of $X \setminus \{x\}$ containing a if $x \neq a$. R_x is defined similarly, to be the empty set if $x = b$ and the component of $X \setminus \{x\}$ containing b if $x \neq b$. Therefore for any x, we have $X = L_x \cup \{x\} \cup R_x$. Note that no L_x contains b and no R_x contains a.

Example 2. Let $X = [0, 1]$, $a = 0$, $b = 1$, and define L_x and R_x as above. Then we have:

Claim 5. *The statements $x \in L_y$ and $L_x \subsetneq L_y$ are equivalent.*

Proof. ($\Longrightarrow$) Assume that $x \in L_y$. Since $x \notin L_x$, we have $L_x \neq L_y$. The point $y \neq a$ since $L_a = \emptyset$, giving a contradiction. If $y = b$, we have $L_y = L_b$ which is defined to be the component of $X \setminus \{y\} = X \setminus \{b\}$ which contains a, which is all of $X \setminus \{b\}$ since b is an exceptional point, so $X \setminus \{b\}$ is connected. That is, certainly $L_x \subseteq L_y$. If $x = a$, then $L_x = \emptyset \subseteq L_y$. If $x = b$, we have a contradiction since then $b \in L_y$, which is impossible.

Excluding these cases, it follows from Claim 2 that either we have $L_x \subseteq L_y$ or $R_x \cup \{x\} \subseteq L_y$. Since L_y does not contain b, and R_x does, it must be that $L_x \subsetneq L_y$.

($\Longleftarrow$) Assume that $L_x \subsetneq L_y$. Excluding the edge cases from before, we have

$$L_x \cup \{x\} = \overline{L_x} \subseteq \overline{L_y} = L_y \cup \{y\}.$$

Since $x \neq y$ because $L_x \neq L_y$, we have $x \in L_y$.

Similarly $x \in R_y$ and $R_x \subsetneq R_y$ are equivalent. □

For any $x, y \in X$, we define $x \prec y$ iff $x \in L_y$. Thus $a \prec x$ unless $x = a$ and $x \prec b$ unless $x = b$. Further, $\nexists x \in X$ such that $x \prec a$ or $b \prec x$.

Claim 6. *The relation $\prec$ is a total ordering.*

Proof. (a) Assume there is $x \in X$ such that $x \prec x$. This means that $x \in L_x$, that is, x is contained in the component of $X \setminus \{x\}$ which contains a, which is a contradiction.

(b) Let $x, y, z \in X$ such that $x \prec y$ and $y \prec z$. Then we have that $x \in L_y$ and $y \in L_z$. By Claim 5, we have equivalently that $L_x \subsetneq L_y$ and $L_y \subsetneq L_z$, so certainly $L_x \subsetneq L_z$. Then again by Claim 5, we have $x \in L_z$, so $x \prec z$.

(c) Let $x, y \in X$ with $x \neq y$. Assume that $x \not\prec y$. That is, $x \notin L_y$. If $x \notin L_y$ and $x \neq y$, then $x \in R_y$. By Claim 5, we have $R_x \subsetneq R_y$. Taking complements, we have $L_y \cup \{y\} \subseteq L_x \cup \{x\}$, and since $y \neq x$, we have $y \in L_x$. Therefore $y \prec x$.

The proof of the opposite direction is similar. □

The set given by $\{x \in X \mid x \prec p\}$ for any $p \in X$ is L_p and therefore an open set. Similarly, $\{x \in X \mid p \prec x\} = R_p$ for any $p \in X$. The set $\{x \in X \mid p \prec x \prec q\}$ is the intersection of two of these open sets, and is thus open as well. We denote this set by $\prec p, q \succ$.

Claim 7. *If $p \prec q$, the set $\prec p, q \succ \neq \emptyset$.*

Proof. Suppose that $p \prec q$ and $\prec p, q \succ = \emptyset$. That is, every point of X must be in either $L_p \cup \{p\}$ or $R_q \cup \{q\}$, so

$$(L_p \cup \{p\}) \cup (R_q \cup \{q\}) = X.$$

We also have

$$(L_p \cup \{p\}) \cap (R_q \cup \{q\}) = \emptyset$$

because

$$(L_p \cap R_q) \cup (L_p \cap \{q\}) \cup (\{p\} \cap R_q) \cup (\{p\} \cap \{q\}) = \emptyset$$

since each set in the union is empty because $p \prec q$. Then we have the partition

$$X = (L_p \cup \{p\}) \mid (R_q \cup \{q\}),$$

which is a contradiction since X is connected. □

Let E_0 be a countable dense set in X not containing a or b, say $E_0 = \{x_1, x_2, \ldots\}$. If $p \prec q$, the nonempty open set $\prec p, q \succ$ contains at least one point of E_0, and in particular there is a point of E_0 between any two points of E_0 since $\prec x_i, x_j \succ$ is nonempty and open.

Let $\{\alpha_1, \alpha_2, \ldots\}$ be any enumeration of the rational points of $(0, 1)$. We construct two sequences, $\{y_1, y_2, \ldots\} \subseteq E_0$ and $\{\beta_1, \beta_2, \ldots\} \subseteq (0, 1) \cap \mathbb{Q}$ as follows:

Let $y_1 = x_1$ and $\beta_1 = \alpha_1$. Suppose that y_n and β_n have been defined for $n \leq k-1$. If k is even, let $y_k = x_m$ where m is the least index for which $x_m \neq y_i$ for $i \leq k-1$. Let $\beta_k = \alpha_m$ for the least index m for which $\alpha_m \neq y_i$ for $i \leq k-1$ *and* if $y_{k-1} \prec y_k$, then $\beta_{k-1} < \beta_k$, or if $y_k \prec y_{k-1}$, then $\beta_k < \beta_{k-1}$. (We do this so that the map we create later is order-preserving). If k is odd, we reverse the order in which we assign the values. That is, let $\beta_k = \alpha_m$ for the least index m for which $\alpha_m \neq \beta_i$ for $i \leq k-1$. Let $y_k = x_m$ for the least index m for which $x_m \neq y_i$ for $i \leq k-1$ *and* if $\beta_{k-1} < \beta_k$, then $y_{k-1} \prec y_k$, or if $\beta_k < \beta_{k-1}$, then $y_k \prec y_{k-1}$. In this way, we define each new element of a sequence based on the order of the element last added to the other sequence.

Example 3. As described above, we let $y_1 = x_1$ and $\beta_1 = \alpha_1$. Since 2 is even, we follow the even definition of $\{y_n\}$ and $\{\beta_n\}$. We let $y_2 = x_2$ since 2 is the least index of $\{x_n\}$ which has not been assigned. Now we have two cases:

(Case 1) $y_1 \prec y_2$. In this case, we choose m to be the least index of the $\{\alpha_n\}$ which hasn't been assigned *and* has the relation $\alpha_1 < \alpha_m$ and define $\beta_2 = \alpha_m$.

(Case 2) $y_2 \prec y_1$. In this case, we choose m to be the least index of the $\{\alpha_n\}$ which hasn't been assigned *and* has the relation $\alpha_m < \alpha_1$ and define $\beta_2 = \alpha_m$.

Since 3 is odd, we follow the odd definition of $\{y_n\}$ and $\{\beta_n\}$. We let $\beta_3 = \alpha_i$ where $i = 3$ or $i = 2$ depending on whether α_2 has been used or not. Similar to before, we have two cases:

(Case 1) $\beta_2 < \beta_3$. In this case, we choose m to be the least index of the $\{x_n\}$ which hasn't been assigned *and* has the relation $x_2 < x_m$ and define $y_3 = x_m$.

(Case 2) $\beta_3 < \beta_2$. In this case, we choose m to be the least index of the $\{x_n\}$ which hasn't been assigned *and* has the relation $x_m < x_2$ and define $y_3 = x_m$.

We continue in this fashion to construct $\{y_n\}$ and $\{\beta_n\}$.

Certainly we have that each x_i appears as a y_j once and only once because of the definition of the sequence $\{y_1, y_2, \ldots\}$ and similarly for α_i appearing once and only once as a β_j. Another consequence of the way we defined this sequence is that if we define the map f by $f(y_i) = \beta_i$, then this map is order preserving. Thus f is a one-to-one, order preserving map from E_0 to the rational points of $(0, 1)$.

We now proceed in a fashion similar to Dedekind's construction of the real numbers.

Claim 8. *Let Λ be a section of E_0, and K the set of points of X not followed by any point of Λ. Then K has a first point.*

Proof. K cannot be empty because it contains b (recall that $\forall x \in X$ with $x \neq b$, $x \prec b$). If $K = X$, then we must have that a is the first point by a similar reasoning. If we exclude these cases, we can consider $X \setminus K$, which is an open set since if we have $x \in X \setminus K$, since Λ has no last element, we can find a $y \in \Lambda$ such that $x \prec y$, the set $\prec a, y \succ$ is an open neighborhood of x such that $\prec a, y \succ \subseteq X \setminus K$.

If K has no first element, then it is open as well since if $x \in K$, then there is a point $y \in K$ with $y \prec x$ and $z \prec y$ for all $z \in \Lambda$. Then $\prec y, b \succ$ is an open neighborhood of x such that $\prec y, b \succ \subseteq K$. Then this implies that $X = K \mid (X \setminus K)$ is a partition of X. □

We have now shown that each section of E_0 corresponds to a unique point of X, which we call the *point determined by the section.*

Claim 9. *The points determined by two different sections are different.*

Proof. Let Λ_1 and Λ_2 be two sections of E_0 with $\Lambda_1 \neq \Lambda_2$, Λ_1 determining the point x_1, and Λ_2 determining the point x_2. Clearly if $x \in \Lambda_1$ and $x \notin \Lambda_2$, then $x \prec x_1$, but $x \not\prec x_2$, so $x_2 \prec x$ and since $\prec$ is a total order, we have that $x_2 \prec x_1$, so $x_2 \neq x_1$. □

Every point of $x \in X$ is determined by at least one section, namely

$$\Lambda = \{y \in E_0 \mid y \prec x\}.$$

Therefore we have set up a one-to-one correspondence between the sections of E_0 and the points of X.

The map $f : E_0 \to \mathbb{Q} \cap (0, 1)$ which we constructed earlier can now be extended to all points of X in the following way: If $x \in X$ and Λ_x is a section of E_0 determining it, then $f(\Lambda_x)$ is the left subset of a Dedekind cut corresponding to some real number y since f is order-preserving. Then we have $x \mapsto y$. Note that this agrees with the old definition of f since if $y_i \in E_0$, then

$$\Lambda_{y_i} = \{z \in E_0 \mid z \prec y_i\}.$$

$f(\Lambda_{y_i}) = (0, \beta_i)$, so $f(y_i) = \beta_i$.

Further note that for $x = a$, we have $\Lambda_a = \emptyset$, and thus $f(\Lambda_a) = \emptyset$. Thus $f(a) = 0$. Similarly, for $x = b$, we have $f(b) = 1$.

Claim 10. *The mapping f is order-preserving, i.e. if $x, y \in X$ with $x \prec y$, then $f(x) < f(y)$.*

Proof. If $x, y \in E_0$, this follows from the original definition of f, which is still valid. If $x \in E_0$ but $y \notin E_0$, $x \in \Lambda_y$ and thus $f(x) \in f(\Lambda_y)$, the section of the rationals determining $f(y)$. Hence $f(x) < f(y)$. If $y \in E_0$ and $x \notin E_0$, we have that $y \notin \Lambda_x$, so $f(y) \notin f(\Lambda_x)$ so $f(x) < f(y)$. If $x, y \notin E_0$, then $\exists z \in E_0$ such that $x \prec z \prec y$. By the above cases, we have $f(x) < f(z)$ and $f(z) < f(y)$, so $f(x) < f(y)$. □

Claim 11. *f is continuous.*

To prove this, we show that if G is an open set in $[0, 1]$, then $f^{-1}(G)$ is open in X. It is sufficient to show that open intervals in $[0, 1]$ (which are the components of G) are mapped to by open sets in X. Let $c, d \in X$ with $f(c) = \gamma$ and $f(d) = \delta$. The typical open intervals in $[0, 1]$ are $[0, \gamma)$, (γ, δ), and $(\delta, 1]$. These are mapped to from the sets $\prec a, c \succ$, $\prec c, d \succ$, and $\prec d, b \succ$, respectively, all open in X.

Thus f is continuous, and since X is compact, f^{-1} is continuous. Therefore we have a homeomorphism between X, a continuum with two or fewer cut-points, and $[0, 1]$. □

Corollary 1. *The open arc is then a space Y with the following properties:*

(a) Y is separable,

(b) Y is connected and locally connected, and

(c) if x is any point, $Y \setminus \{x\}$ has exactly two components.

4 Characterization of the Circle

Theorem 3. *A continuum whose connection is destroyed by the removal of two arbitrary points is a simple closed curve.*

Proof. Let x, y be two points that are not cut points of the space S. Then we have the partition $S \setminus \{x, y\} = P_1 \mid P_2$.

Claim 12. $\overline{P_1} = P_1 \cup \{x, y\}$ *and* $\overline{P_2} = P_2 \cup \{x, y\}$.

Proof. The proof is similar to that given for Theorem 1. Since P_1 is closed in $S \setminus \{x, y\}$, we have

$$P_1 = \overline{P_1} \cap (S \setminus \{x, y\}) = \overline{P_1} \setminus \{x, y\}$$

thus $\overline{P_1} \subseteq P_1 \cup \{x, y\}$. Similarly we have $\overline{P_2} \subseteq P_2 \cup \{x, y\}$. Since $S \setminus \{x\}$ is connected and y is a cut point of it, by Theorem 1, we have that $y \in \overline{P_1}$ and $y \in \overline{P_2}$. Similarly, we get that $x \in \overline{P_1}$ and $x \in \overline{P_2}$, so we have $\overline{P_1} = P_1 \cup \{x, y\}$ and $\overline{P_2} = P_2 \cup \{x, y\}$. □

Lemma 2. *If C is a connected set, then $\overline{C}$ is connected.*

Proof. Assume that $\overline{C}$ is disconnected. Then we have the partition $\overline{C} = P \mid Q$. Then P and Q are both clopen sets in $\overline{C}$, so $C \cap P$ and $C \cap Q$ are clopen sets in C. Since $P \cap Q = \emptyset$, we have

$$(C \cap P) \cap (C \cap Q) = C \cap (P \cap Q) = \emptyset.$$

Since $P \cup Q = \overline{C}$, we have

$$(C \cap P) \cup (C \cap Q) = C \cap (P \cup Q) = C \cap \overline{C} = C.$$

Since $P \neq \emptyset$ and $Q \neq \emptyset$ are open sets in $\overline{C}$ and C is dense in $\overline{C}$, we have that $C \cap P \neq \emptyset$ and $C \cap Q \neq \emptyset$. Thus we have the partition

$$C = (C \cap P) \mid (C \cap Q),$$

which contradicts the fact that C is assumed to be connected. □

Claim 13. $\overline{P_1}$ *and* $\overline{P_2}$ *are connected.*

Proof. The proof is similar to that given for Claim 1. Suppose that $P_1 \cup \{x\}$ is disconnected. Then we have the partition $P_1 \cup \{x\} = H_1 \mid H_2$. Assume without loss of generality that $x \in H_1$. The sets $P_1 \cup \{x\} = \overline{P_1} \setminus \{y\}$ and $P_2 \cup \{x\} = \overline{P_2} \setminus \{y\}$ are closed in $S \setminus \{y\}$, hence H_1 and H_2 are also closed in $S \setminus \{y\}$. Then we have the partition

$$S \setminus \{y\} = H_2 \mid (H_1 \cup (P_2 \cup \{x\})),$$

which contradicts the assumption that y is not a cut point of S. Thus $P_1 \cup \{x\}$ is connected, and by Lemma 2, its closure, $\overline{P_1} = \overline{P_1 \cup \{x\}}$ is also connected. The proof is similar to show that $\overline{P_2}$ is connected. □

Lemma 3. *If C_1 and C_2 are two connected sets and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is connected.*

Proof. Suppose that $C_1 \cup C_2$ is disconnected. Then we have the partition

$$C_1 \cup C_2 = P \mid Q.$$

Since $C_1 \cap C_2 \neq \emptyset$, we can take $x \in C_1 \cap C_2$. By Lemma 1, we must have that $C_1 \subseteq P$ or $C_1 \subseteq Q$. Similarly, we have that $C_2 \subseteq P$ or $C_2 \subseteq Q$. We have a few cases:

(Case 1) $C_1 \subseteq P$ and $C_2 \subseteq P$. Then $C_1 \cup C_2 \subseteq P$, so $Q = \emptyset$, a contradiction.

(Case 2) $C_1 \subseteq P$ and $C_2 \subseteq Q$. Then $C_1 \cap C_2 \subseteq P \cap Q = \emptyset$, a contradiction.

The other two cases are similar. Therefore in every case, we reach a contradiction, so we must have that $C_1 \cup C_2$ is connected. □

Claim 14. *Every point in P_1 is a cut point of $\overline{P_1}$ and every point in P_2 is a cut point of $\overline{P_2}$.*

Proof. Suppose that $u \in P_1$ and $\overline{P_1} \setminus \{u\}$ is connected. Then if $v \in P_2$, $\overline{P_2} \setminus \{v\}$ cannot be connected since otherwise

$$\left(\overline{P_1} \setminus \{u\}\right) \cup \left(\overline{P_2} \setminus \{v\}\right) = S \setminus \{u, v\}$$

would be connected by Lemma 3. This is a contradiction since we assume that the removal of any two points renders the set disconnected. □

From Claims 12, 13, and 14, it follows that both $\overline{P_1}$ and $\overline{P_2}$ are closed arcs with endpoints x and y. Hence S is the union of two arcs with the same endpoints and no other common point, that is, a simple closed curve. □

5 References

[1] M. H. A. Newman. *Elements of the Topology of Plane Sets of Points*, pages 71-100. Cambridge, 1964.