TOPOLOGICAL VECTOR SPACES[1]

1. Definitions and basic facts.

A *topological vector space* (TVS) X is a (real) vector space with a Hausdorff topology τ for which the vector space operations are continuous:

$$R \times X \to X, (c, x) \mapsto cx \quad X \times X \to X, (x, y) \mapsto x + y.$$

In particular, translations and homotheties are homeomorphisms of X.

A neighborhood U of 0 is *balanced* if $cU \subset U$ for any $c \in R$ with $|c| \leq 1$. Any neighborhood V of 0 contains a balanced one: by continuity, one may find $\delta > 0$ and $W \subset V$ (open) so that $cW \subset V$ if $|c| < \delta$. Then:

$$U = \bigcup_{|c|<\delta} cW$$

is a balanced open neighborhood of 0 contained in V.

Similarly, if U is a convex nbd of 0, we may find $A \subset U$, a *balanced* convex nbd of 0. Just let $A = U \cap (-U)$, an open, convex (intersection of convex), symmetric nbd of 0 contained in U, and also balanced: if $inR, |c| \leq 1$:

$$cA = |c|U \cap (-|c|U) \subset U \cap (-U),$$

since $cU \subset U$ for $0 < c < 1$ by convexity.

A set $A \subset X$ is *bounded* if for any neighborhood V of 0 there exists $t > 0$ so that $A \subset tV$.

A) For any $p \in X, C \subset X$ closed with $p \notin C$, we may find a neighborhood V of 0 so that:

$$(p + V) \cap (C + V) \neq \emptyset.$$

Thus, any TVS is *regular*.

Proof. First, for any W nbd of 0, there exists $U \subset W$ symmetric nbd of 0, ($U = -U$) so that $U + U \subset W$. To see this, note that (by continuity of addition at 0) we may find V_1, V_2 nbds. of 0, so that $V_1 + V_2 \subset W$. Then let:

$$U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2).$$

Thus we may find V symmetric nbd of 0 so that $p + V + V \subset X \setminus C$, or $(p + V + V) \cap C = \emptyset$. Then also $(p + V) \cap (C + V) = \emptyset$.

[1] reference: W. Rudin, Functional Analysis, Chapter 1

B) (i) If V is a nbd of 0 and $r_n \uparrow \infty$, then $\bigcup_n r_n V = X$.

Let $x \in X$. The set $\{\alpha \in R; \alpha x \in V\}$ is open and contains 0, hence contains $1/r_n$ for all large enough n. Thus $\frac{1}{r_n} x \in V$, or $x \in r_n V$.

(ii) Compact subsets $K \subset X$ are bounded.

Proof. Let V be a nbd of 0, and find $U \subset V$ balanced. Then $K \subset \bigcup_{n \geq 1} nU$, so

$$K \subset n_1 U \cup \ldots \cup n_s U, \quad n_j \text{ increasing} .$$

Since U is balanced, this implies $K \subset n_s U$, so $K \subset tU \subset tV$ for $t > n_s$.

(iii) If $V \subset X$ is a bounded neighborhood of 0 and $\delta_n \downarrow 0$, then $\{\delta_n V\}_{n \geq 1}$ is a countable local basis (at 0).

Thus: X locally bounded (i.e. 0 has a bounded neighborhood)$\Rightarrow$ X is first countable.

Proof. Let U be a nbd. of 0. Then $V \subset \frac{1}{\delta_n} U$ for $n \geq n_0$, or $U \supset \delta_n V$.

C) *Metrization.* A locally bounded, separable TVS is metrizable.

Indeed, as seen in A and B(iii) above, X is regular and first countable, hence also second countable (given separable), and the Urysohn metrization theorem gives the conclusion.

Remark. Indeed 'separable' is not needed here; a direct construction of the metric is given in [Rudin, p.18]. The metric is translation-invariant, with balanced balls at the origin. An important explicit construction in a special case is given later. (When τ is defined by a countable collection of seminorms.)

Note on the different notions of 'bounded set'. On a normed vector space, both notions coincide. But not for a general metric. For instance, the metrics d and $d_1 = d/(1+d)$ define the same topology, and X is d_1-bounded. But X cannot be bounded in the general TVS sense.

Indeed, if $x \neq 0$ the set $E = \{nx; n = 1, 2, \ldots\}$ is not bounded. There is a nbd V of 0 with $x \notin V$. Thus $nx \notin nV$, so for no $t > 0$ is it the case that $E \subset tV$.

2. Local compactness.

Definition. A TVS is *locally compact* of 0 has a nbd V with $\overline{V}$ compact. Thus V is bounded and $\{2^{-n} V\}$ is a local basis at 0.

Lemma. In any TVS, finite-dimensional subspaces are closed.

Theorem. A locally compact TVS is finite dimensional.

Idea of proof. [Rudin, p. 17] Let V be a nbd of 0 with compact closure. We know V is bounded, and the sets $2^{-n}V, n \geq 1$ form a local basis for X. By compactness, we may find $x_1, x_2, \ldots x_m$ in X so that:

$$\bar{V} \subset (x_1 + \frac{1}{2}V) \cup \ldots \cup (x_m + \frac{1}{2}V).$$

Let $Y \subset X$ be the subspace spanned by $x_1, \ldots, x_m$. Y is finite-dimensional, hence closed (by the Lemma.) And we know $V \subset Y + \frac{1}{2}V$, hence $\frac{1}{2}V \subset Y + \frac{1}{4}V$ and:

$$V \subset Y + \frac{1}{2}V \subset Y + Y\frac{1}{4}V = Y + \frac{1}{4}V,$$

and proceeding we find:

$$V \subset \bigcap_{n \geq 1}(Y + 2^{-n}V),$$

and since $\{2^{-n}V\}$ is a local basis, the set on the right is $\bar{Y}$, or Y. (Here we use the fact that, for a set $A \subset X$, the closure is:

$$\bar{A} = \bigcap (A + V); V \text{ a nbd of } 0.$$

This since $x \in \bar{A}$ iff $(x + V) \cap A \neq \emptyset$ iff $x \in A - V$ for all nbds. V of 0.)

Thus $nV \subset Y$ for $n \geq 1$, so $X = Y$.

Corollary. A locally bounded TVS with the Heine-Borel property is finite dimensional. (Follows since $\overline{V}$ is bounded and closed, hence compact.)

3. Seminorms, local convexity and normability.

Definition. A *seminorm* on a vector space X is a function $p : X \to R_+$ satisfying:

$$p(x + y) \leq p(x) + p(y), \quad p(cx) = |c|p(x) \quad p(0) = 0.$$

It is easy to see that $N = \{x \in X; p(x) = 0\}$ is a subspace. If $N = \{0\}$, p is a norm. Also:

$$|p(x) - p(y)| \leq p(x - y)$$

for any seminorm.

The set $B = \{x; p(x) < 1\}$ contains 0 and is convex, balanced and *absorbing*: $\bigcup_{t>0} tB = X$ (since $p(x) < t$ implies $\frac{x}{t} \in B$, or $x \in tB$), but B may be unbounded.

Conversely, given a convex, balanced, absorbing set $A \subset X$ containing 0, define its *Minkowski functional* by:

$$\mu_A(x) = \inf\{t > 0; \frac{x}{t} \in A\}.$$

Lemma. $\mu_A(x)$ is a seminorm on X. If A is bounded, μ_A is a norm.

Proof. (i) For $\lambda > 0$, we have (setting $t = \lambda s$):

$$\mu_A(\lambda x) = \inf\{t > 0; \frac{\lambda x}{t} \in A\} = \inf\{\lambda s; s > 0, \frac{x}{s} \in A\} = \lambda \inf\{s > 0; \frac{x}{s} \in A\} = \lambda \mu_A(x).$$

And similarly, now setting $s = -t$:

$$\mu_A(-x) = \inf\{t > 0; \frac{(-x)}{t} \in A\} = \inf\{-s > 0; \frac{x}{s} \in A\} = \inf\{s > 0; \frac{x}{s} \in A\} = \mu_A(x).$$

Thus $\mu_A(\lambda x) = |\lambda| \mu_A(x)$, for all $\lambda \in R$.

(ii) Given $x, y \in X$ we have, for any $\epsilon > 0$:

$$\frac{x}{\mu_A(x) + \epsilon} \in A, \quad \frac{y}{\mu_A(y) + \epsilon} \in A.$$

But clearly:

$$\frac{x + y}{\mu_A(x) + \mu_A(y) + 2\epsilon} = \lambda_1 \frac{x}{\mu_A(x) + \epsilon} + \lambda_2 \frac{y}{\mu_A(y) + \epsilon},$$

where

$$\lambda_1 = \frac{\mu_A(x) + \epsilon}{\mu_A(x) + \mu_A(y) + 2\epsilon}, \lambda_2 = \frac{p(y) + \epsilon}{\mu_A(x) + \mu_A(y) + 2\epsilon}.$$

Since $\lambda_1 + \lambda_2 = 1$, convexity of A implies $\frac{x+y}{\mu_A(x)+\mu_A(y)+2\epsilon} \in A$. Since $\epsilon > 0$ is arbitrary, this shows:

$$\mu_A(x + y) \leq \mu_A(x) + \mu_A(y),$$

proving the claim.

(iii) Note $\mu_A(x) = 0$ iff $tx \in A \forall t > 0$. If $x \neq 0$, we may find V nbd of 0 with $x \notin V$. Then $tx \notin tV$ for any $t > 0$, in particular $tx \notin A$ for t large enough (since A is bounded.) Thus $\mu_A(x) > 0$. This shows μ_A is a norm if A is bounded.

If p is a seminorm on X and $B = \{x; p(x) < 1\}$, then $\mu_B = p$. Indeed let $x \in X$. If $s > p(x)$, we have $p(\frac{x}{s}) < 1$, or $\frac{x}{s} \in B$, so $s \geq \mu_B(x)$. Thus

$p(x) \geq \mu_B(x)$. And if $0 < s \leq p(x)$, $p(\frac{x}{s}) \geq 1$, so $\frac{x}{s} \notin B$, and $s \leq \mu_B(x)$. This shows $p(x) \leq \mu_B(x)$. We conclude $\mu_B(x) = p(x)$: any seminorm is the Minkowski functional of its 'unit ball'.

Normability Theorem. (Kolmogorov 1932) Let X be a TVS. Then X is normable (topology induced by a norm) if and only if 0 has a bounded, convex neighborhood V.

Prof. Necessity is clear. To prove sufficiency, note we may assume V is balanced. Thus μ_V is a norm on X. Since V is open, we have:

$$B(1) = \{x; \mu_V(x) < 1\} = V \text{ and for each } r > 0 : B(r) = \{x; \mu_V(x) < r\} = rV.$$

On the other hand, for any sequence $r_n \downarrow 0$ the sets $B(r_n) = r_n V$ form a local basis (as seen in 1.B(iii)). Thus the norm topology and the original topology of X coincide.

4. Locally convex first countable spaces are metrizable via a countable family of seminorms.

Example. Let $X = C(R^n)$ be the space of continuous functions, with the topology τ of uniform convergence on compact subsets. A basis at 0 is given by:

$$V(K, M) = \{f \in X; f(K) \subset [-M, M]\}; \quad K \subset R^n \text{ compact }, M > 0.$$

Let $\{K_i\}_{i \geq 1}$ be an exhaustion of R^n by closed balls at 0. Consider the seminorms on X:

$$p_i(f) = \sup\{|f|(x)|; x \in K_i\}.$$

This family of seminorms is *separating*: if $f \not\equiv 0$ on R^n, then $p_i(f) \neq 0$ for some i. The sets:

$$V(p_i, j) = \{f \in X; p_i(f) < \frac{1}{j}\}$$

define a countable basis at 0 for the topology τ. (Note that these sets are convex and balanced.) And the expression:

$$d(f, g) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(f - g)}{1 + p_i(f - g)|}$$

defines a metric on X (inducing τ), translation invariant and with balanced balls at 0.

Let (f_n) be a Cauchy sequence in (X, d). Then $p_i(f_n - f_m) \to 0$ as $n, m \to \infty$, so (f_n) is Cauchy on each K_i (with the sup metric), and a diagonal argument gives $f_n \to f$ uniformly con compact sets, where $f \in X$. It is easy to see that $d(f, f_n) \to 0$. Thus the metric d is *complete.*

This example generalizes, as follows:

Proposition. Let $\{V_i\}_{i \geq 1}$ be a countable local basis of balanced convex sets for a TVS X; let p_i be the Minkowski functional of V_i. Then $\{p_i\}$ is a separating family of continuous seminorms on X.

Theorem. Let $\{p_i\}_{i \geq 1}$ be a countable separating family of seminorms on the vector space X. Consider the sets:

$$V(p_i, n) = \{x \in X; p_i(x) < \frac{1}{n}\}.$$

(i)Taken as a subbasis, these sets define a TVS topology (Hausdorff, vector space operations are continuous) τ on X , turning X into a locally convex first countable TVS;

(ii) The seminorms p_i are continuous in this topology;

(iii) A set $E \subset X$ is bounded (w.r.t. τ) if an only if each p_i is bounded on E.

(iv) This topology is metrizable, via:

$$d(x, y) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(x-y)}{1 + p_i(x-y)}.$$

This is a translation-invariant metric, with balanced balls at 0.

Corollary. A locally convex first countable TVS is metrizable.

5. Three examples.

A) $X = C(R^n)$ with the topology of uniform convergence on compact sets: a completely metrizable, locally convex space (i.e. a *Frechet space.*) But not locally bounded, hence not normable.

B) $X = C^\infty(R^n)$, with the topology of uniform convergence on compact sets of derivatives up to a finite order: Frechet space with the Heine-Borel property (closed bounded sets are compact). Not locally bounded, hence not normable.

C) On $X = \mathcal{F}(R^n)$ (all real-valued functions on R^n), pointwise convergence corresponds to a TVS that is not first countable, hence not metrizable.