1. PRODUCT TOPOLOGY.

L: arbitrary index set, $(X_\lambda)_{\lambda\in L}$ topological spaces.

$$X = \prod_{\lambda\in L} X_\lambda = \{x : L \to \bigcup_{\lambda\in L} X_\lambda; x(\lambda) = x_\lambda \in X_\lambda \forall\lambda.\}$$

(The fact that $X \neq \emptyset$ if all $X_\lambda \neq \emptyset$ depends on the Axiom of Choice.)

Let $p_\lambda : X \to X_\lambda$ be the canonical projection, $p_\lambda(x) = x_\lambda$. We seek the coarsest topology on X(as few open sets as possible) making all p_λ continuous.

Product topology:
Subbasis: $p_\lambda^{-1}(U)$, where $\lambda \in L$ and $U \subset X_\lambda$ is open. Thus a *basis* is given by sets of the form:

$$V(\lambda_1, \ldots, \lambda_n, U_1, \ldots, U_n) = \{x \in X; x_{\lambda_i} \in U_i, i = 1, \ldots, n\},$$

where $n \geq 1$ and $U_i \subset X_{\lambda_i}$ open. (Thus only finitely many coordinates are constrained.)

Example. If all $X_\lambda = Y$, the same topological space, the product space in question is the space $Y^L = \mathcal{F}(L; Y)$ of all functions from L to Y, with the topology of pointwise convergence.

Proposition. A sequence $x^n = (x^n_\lambda)_{\lambda\in L}$ in X converges to $a = (a_\lambda)_{\lambda\in L}$ iff $x^n_\lambda \to a_\lambda, \forall\lambda$.

This follows easily from the fact that a local basis at $a \in X$ is given by:

$$V(a; \lambda_1, \ldots, \lambda_n, U_1, \ldots, U_n) = \{x \in X; x_{\lambda_i} \in U_i, i = 1, \ldots, n\},$$

where $U_i \subset Y$ is a neighborhood of a_{λ_i}. And $x_n \to a$ iff any such neighborhood eventually contains all x^n iff each U_i eventually contains all $x^n_{\lambda_i}$ iff $x_{\lambda_i} \to a_{\lambda_i}$.

Properties of the product topology.
1. If L is countable and each X_λ is first-countable, then X is first-countable (easy).

Conversely, if X is first-countable L must be countable, and each X_λ first-countable (provided no X_λ has the trivial topology.)

For example $\mathcal{F}(\mathbb{R}; \mathbb{R})$ is not first-countable (and hence is a non-metrizable TVS.) And $\mathcal{F}(\mathbb{R}; [0, 1])$ is non-metrizable and *compact* (by Tychonoff's theorem.) Here we take the topology of pointwise convergence in both cases.

To see this, note that a fundamental system of nbds of $f : \mathbb{R} \to [0,1]$ in $[0,1]^{\mathbb{R}}$ is given by:

$$V(f, t_1, \ldots, t_k; \epsilon) = \{g : \mathbb{R} \to [0,1]; |g(t_i) - f(t_i)| < \epsilon, \forall i\}.$$

If there is a countable local basis at f, $\mathcal{B}(f) = \{V_1, \ldots, V_n\}$, then there is also a local basis of the form: $A_n = V(f; t_{n1}, \ldots, t_{nk}, \epsilon_n)$, such that $f \in A_n \subset V_n$. The $\{t_{ni}\}$ form a countable set, so there exists $t_0 \in \mathbb{R}$ different from all the t_{ni}. Then $A_0 = V(f, t_0, 1)$ is not contained in any A_n. (Since the value at t_0 is unrestricted for functions in A_n, so $\exists g \in A_n; |g(t_0) - f(t_0)| \geq 1.\}$). Contradiction.

A similar argument shows L is countable whenever X is first-countable. And the same statements hold for 'second countable'.

2. *Metrizabilty.* Let (X_λ be nontrivial metric spaces (i.e. each with at least two points.) Then their product X is metrizable iff the index set L is countable. In that case, a metric inducing the product topology is:

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d(x_i, y_i)}{1 + d(x_i, y_i)}.$$

The product space $\prod_{i \geq 1} X_i$ is complete iff each X_i is.

3. *Separation properties.*

a) X is Hausdorff iff each X_λ is Hausdorff. **Exercise 1.**

b) X is regular iff each X_λ is regular.

c) X is completely regular (see below) iff each X_λ is.

d) The product of two normal spaces is not always normal. But if X (arbitrary product) is normal, then each X_λ is.

4. *Connectedness.* X is connected iff each X_λ is; same for path-connected.

2. TYCHONOFF's THEOREM

Finite Intersection Property. A collection $\mathcal{F}$ of subsets of a set X has the FIP if any intersection of finitely many sets in $\mathcal{F}$ is non-empty.

It is easy to see (by complementation) that X is a compact topological space iff any collection $\mathcal{F}$ of *closed* subsets of X with the FIP has non-empty intersection:

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset.$$

Lemma. Let X be a set, $\mathcal{F}$ a collection of subsets with the FIP. Then there exists a *maximal* collection $\mathcal{M}$ of subsets with the FIP, containing $\mathcal{F}$.

The proof follows from Zorn's lemma. (Just order the families $\mathcal{F}$ of subsets with the FIP by inclusion. It is easy to check that each chain of such families has an upper bound, given by their union.)

Note that since $\mathcal{M}$ is maximal, if $S \subset X$ intersects every set in $\mathcal{M}$, then $S \in \mathcal{M}$. In fact $S \in \mathcal{M}$ iff S intersects every $M \in \mathcal{M}$.

Also: any finite intersection $F_1 \cap \ldots \cap F_n$ of sets in $\mathcal{M}$ is itself in $\mathcal{M}$ (since it intersects any set in $\mathcal{M}$, by the FIP.)

Tychonoff's theorem. $X = \prod_{\lambda \in L} X_\lambda$ is compact iff each X_λ is compact.

Proof. One direction is clear, since $X_\lambda = p_\lambda(X)$, each p_λ is continuous, and X is compact.

On the other hand, suppose all X_λ are compact, and let $\mathcal{F}$ be a family of closed subsets of X with the FIP. Our goal is to find an $x \in X$ that is in each $F \in \mathcal{F}$. .

Let $\mathcal{M}$ be maximal for the FIP, containing $\mathcal{F}$. (Note the sets in $M \in \mathcal{M}$ need not be closed.)

For each $\lambda \in L$, the collection $\{p_\lambda(M); M \in \mathcal{M}\}$ of subsets of X_λ has the FIP, since

$$p_\lambda(M_1) \cap \ldots \cap p_\lambda(M_n) \supset p_\lambda(M_1 \cap \ldots \cap M_n) \neq \emptyset.$$

So their closures $\overline{p_\lambda(M)}$, $M \in \mathcal{M}$, are a collection of closed subsets of X_λ with the FIP. Since the X_λ are compact, we may find $x_\lambda \in X_\lambda$ so that $x_\lambda \in \overline{p_\lambda(M)}$, $\forall M \in \mathcal{M}$. (Note that this uses the Axiom of Choice!)

Let $x = (x_\lambda) \in X$. We claim that $\forall M \in \mathcal{M}$, $x \in \overline{M}$. (In particular $(\forall F \in \mathcal{F}) x \in F$, as we wish to show.) That is, we want to show that ($\forall V$ nbd of x in X)$(\forall M \in \mathcal{M})(V \cap M \neq \emptyset)$. And it's clearly enough to show this when V is a basic neighborhood of x.

Let $M \in \mathcal{M}$, let $p_\lambda^{-1}(U)$ be a subbasic neighborhood of x; so $U \subset X_\lambda$ is a nbd of x_λ. Thus, since $x_\lambda \in \overline{p_\lambda(M)}$, we have:

$$U \cap p_\lambda(M) \neq \emptyset, \text{ so } p_\lambda^{-1}(U) \cap M \neq \emptyset.$$

Since $p_\lambda^{-1}(U)$ intersects each set $M \in \mathcal{M}$, by maximality we have $p_\lambda^{-1}(U) \in \mathcal{M}$. Since this is true for an arbitrary subbasic neighborhood of x, it is also true for a finite intersection of them; so any basic neighborhood of x is in the maximal collection $\mathcal{M}$, and in particular intersects each $M \in \mathcal{M}$, as we wished to show.

3. COMPACTIFICATIONS

A *compactification* of a topological space X s a topological embedding (homeomorphism onto its image) $\phi : X \to P$, where P is a compact Hausdorff space and $\phi(X)$ is dense in P.

Ideally, given a family $\mathcal{F}$ of bounded continuous functions on X, we'd like all $f \in \mathcal{F}$ to extend continuously to $\bar{f} \in C(P)$, with the same bound; in the sense that, on X, $f = \bar{f} \circ \phi$. It is enough to consider functions taking values in the interval $I = [0, 1]$.

3.1 Locally compact spaces and Alexandroff compactification.

Recall X (Hausdorff) is *locally compact* if any point admits a neighborhood basis of relatively compact open sets (i.e. with compact closure.)

Some properties. (i) Locally compact spaces are completely regular (see below).

(ii) A product $X = \prod X_\lambda$ is locally compact iff each X_λ is locally compact, and all but finitely many are compact.

(iii) If $p : X \to Y$ is a proper map (preimage of compact is compact), X is locally compact iff Y is.

(iv) Locally compact subsets A of a Hausdorff space X are *locally closed*: this means A is relatively closed within a set U open in X; equivalently, $A = U \cap F$, where F is closed in X (think of the open interval $(0, 1)$ as a subset of $\mathbb{R} \times \{0\}$ in $\mathbb{R}^2$.) If X is locally compact, any subset of this form is a locally compact space.

Item (iv) implies that in any compactification (ϕ, P) of X, $\phi(X)$ is open in P. The reason is $\phi(X)$ is dense in P, and $\phi(X) = U \cap F$, with U open and F closed in P. In particular $\overline{\phi(S)} \subset F$, so $\phi(S) = F$, so $\phi(X) = U$.

Locally compact spaces (and only those) can be compactified by adding one point, as follows: let $\hat{X} = X \sqcup \{\infty\}$, where $\infty \notin X$. The open sets of $\hat{X}$ are (by definition):

(i) open sets of X;

(ii) sets of the form $(X \setminus K) \sqcup \{\infty\}$, where $K \subset X$ is compact.

Exercise 2. (i) Show this defines a topology in the set $\hat{X}$;

(ii) Show this topology is Hausdorff;

(iii) Show the resulting space is *compact.*

The inclusion $\phi : X \to \hat{X}$ defines the *Alexandroff compactification* of X. It can be shown that it is the unique one-point compactification, up to homeomorphism.

It is natural to ask when $\hat{X}$ is first or second countable, or is metrizable. For "first countable", the main issue is whether X is "countable at infinity" (that is, there is a countable local basis of nbds at ∞ for $\hat{X}$). We have the following:

Proposition 1. A locally compact Hausdorff space is countable at infinity iff it is σ-compact: there exists a sequence (K_n) of compact subsets of X so that $K_n \subset \text{ int } K_{n+1}$ and $\bigcup_{n\geq 1} K_n = X$.

Proposition 2. For a locally compact *metric* space X, the following are equivalent:
(i) X is second countable;
(ii) X is countable at infinity (or σ-compact);
(iii) The Alexandroff compactification $\hat{X}$ is metrizable;

It is natural to ask when a bounded continuous function on a locally compact space X extends continuously to $\hat{X}$. This motivates the following:
Definition. Let X be locally compact Hausdorff. A function $f \in C(X;\mathbb{R})$ is *constant at infinity* if for all $\epsilon > 0$, we may find $K \subset X$ compact so that $p, q \in X \setminus K \Rightarrow |f(p) - f(q)| < \epsilon$.

Exercise 3. (i) Show that this condition implies f is bounded on X, and has a limit at infinity:

$$(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists K \subset X \text{ compact })(\forall x \in X \setminus K)(|f(x) - L| < \epsilon).$$

(ii) Show that $f \in C(X)$ extends continuously to $\hat{X}$ iff f is constant at infinity;
(iii) Show that the family $\mathcal{F} \subset C(X)$ of functions constant at infinity is an algebra vanishing nowhere and separating points, and in fact separates points from closed sets (see definition below.)

3.2 Completely regular spaces and Stone-Cech compactification.

Let $\mathcal{F} \subset C(X)$ be a family of functions with values in the interval $I = [0,1]$. The idea is to find an embedding into $I^{\mathcal{F}}$ (the space of all functions from $\mathcal{F}$ to I), which is compact by Tychonoff's theorem. Points of P are families $(t_f)_{f\in\mathcal{F}}$ of points in I.There is a natural map:

$$\phi : X \to I^{\mathcal{F}}, \quad \phi(x)_f = f(x) \in I, \quad f \in \mathcal{F} \subset C(X;I).$$

The map ϕ is continuous on X, since for each $f \in \mathcal{F}$, $p_f \circ \phi = f$ is continuous on X (where $p_f : P \to I$ is the projection on the f-th. component.)

To establish ϕ is an embedding, we need a condition on $\mathcal{F}$:

Definition. We say $\mathcal{F}$ *separates points from closed sets* if for all $x \in X$, and all $F \subset X$ closed such that $x \notin F$, there exists an $f \in \mathcal{F}$ so that $f(x_0) = 1$ and $f \equiv 0$ on F.

We show that if $\mathcal{F}$ satisfies this condition, ϕ is an embedding. First, let $x \neq y$ in X, and let U be a nbd of x so that $y \notin U$. Then we may find $f \in \mathcal{F}$ so that $f(x) = 1$ and $f \equiv 0$ on U^c, in particular $f(y) = 0$. Thus $\phi(x)_f = f(x) \neq f(y) = \phi(y)_f$; so $\phi(x) \neq \phi(y)$, showing ϕ is injective.

We show ϕ is open from X to $\phi(X)$ (with the induced topology from the product topology in $I^{\mathcal{F}}$), in three steps.

i) As f varies in $\mathcal{F}$, the open sets $V_f = \{x \in X; f(x) > 0\}$ form a basis of X: indeed given $x_0 \in X$ and a nbd U of x_0, we may find $f \in \mathcal{F}$ so that $f(x_0) = 1$, $f \equiv 0$ on U^c. But then V_f is a nbd of x_0 and $V_f \subset U$.

ii) For each $f_0 \in \mathcal{F}$, the set $A_{f_0} = \{t = (t_f)_{f \in \mathcal{F}}; t_{f_0} > 0\}$ is open in $I^{\mathcal{F}}$, and indeed is a subbasic open set.

iii) $\phi(V_{f_0}) = A_{f_0} \cap \phi(X)$. It is enough to observe that:

$$y = \phi(x) \in A_{f_0} \Leftrightarrow \phi(x)_{f_0} > 0 \Leftrightarrow f_0(x) > 0 \leftrightarrow x \in V_{f_0}.$$

Now just let $P = \overline{\phi(X)}$ to get a compactification $\phi : X \to P$ with dense image. We have asssociated a compactification of X to each family $\mathcal{F} \subset C(X; I)$ satisfying the separation property. And each $g \in \mathcal{F}$ extends continuously to $\bar{g} \in C(P; [0,1])$. Define:

$$\bar{g} : I^{\mathcal{F}} \to [0,1]; \quad \bar{g}[(t_f)_f] = t_g.$$

Then we see that:

$$(\bar{g} \circ \phi)(x) = \bar{g}[(t_f)_f] = t_g \text{ where } t_f = f(x) \forall f \in \mathcal{F} \text{ so } t_g = g(x);$$

so $\bar{g} \circ \phi = g$.

In particular, if we take the family $\mathcal{C} = C(X; I)$ we get the *Stone-Cech compactification.* The condition for existence is:

Definition. A Hausdorff space X is *completely regular* if the family of all continuous functions from X to I separates points from closed sets: for all $x \in X$, and all $F \subset X$ closed such that $x \notin F$, there exists an $f \in C(X; I)$ so that $f(x_0) = 1$ and $f \equiv 0$ on F.

If X is a completely regular Hausdorff space, the *Stone-Cech compactification* $\beta(X)$ is given by the embedding $\phi : X \to I^{\mathcal{C}}$, $\phi(x) = (t_f)_{f \in \mathcal{C}}$, $t_f = f(x)$; specifically, $\beta(X) = \overline{\phi(X)}$ (closure in $I^{\mathcal{C}}$).

PROPERTIES:

1) (Extension) Any $f : X \to I$ continuous extends to $\bar{f} : \beta(X) \to I$ (continuous), in the sense that $f = \bar{f} \circ \phi$. In fact, this is still true if we replace I by an arbitrary compact Hausdorff space Y.

Remark: If X is metric, $\beta(X)$ is not necessarily metrizable. When it is, it is complete for any metric; but it rarely coincides with the metric completion of X (since ϕ is usually not an isometry.) Also, only uniformly continuous functions on a metric space X extend continuously to its completion, while *all* continuous functions on X extend to $\beta(X)$. For instance, if $X = (0, 1)$ (with the usual metric), $\beta(X)$ is "much bigger" than the metric completion $[0, 1]$.

2) (Maximality) If $\psi : X \to Y$ (compact) is any compactification of X, there exists $F : \beta(X) \to Y$ continuous, such that:

$$\psi = F \circ \phi_X, \text{ where } \phi_X : X \to \beta(X)$$

is the defining embedding.

3) (Functoriality) If $f : X \to Y$ is a continuous map, with X, Y completely regular, there exists $\hat{f} : \beta(X) \to \beta(Y)$ continuous, satisfying:

$$\phi_Y \circ \hat{f} = f \circ \phi_X,$$

where $\phi_X : X \to \beta(X), \phi_Y : Y \to \beta(Y)$ are the defining embeddings.

4) (Uniqueness) Any compactification of X (comletely regular) satisfying either the extension or the maximality properties is homeomorphic to $\beta(X)$ (by a homeomorphism extending the identity on X.)

Remark. At this point it is useful to think about the proof of the Urysohn metrization lemma in this setting. Recall X is a second countable normal space. We take a countable basis of open sets $\{U_n\}_{n\geq 1}$, and given an "admissible pair" (U_m, U_n) (this means $\overline{U_m} \subset U_n$) choose a Urysohn function $f \in C(X; I)$ which is $\equiv 1$ on $\overline{U_m}$, $\equiv 0$ on U_n^c. This defines a countable family $\mathcal{F} \subset C(X; I)$, and it is not hard to see $\mathcal{F}$ separates points from closed sets.

Then we define an embedding $\phi : X \to C$ (the Hilbert cube) exactly as in the construction above. Note C is homeomorphic to the countable product $I^{\mathbb{N}}$. So this embedding defines a compactification of X, with values in the separable compact metric space C. *Question.* Is $\overline{\phi(X)}$ (closure in C) homeomorphic to $\beta(X)$, the Stone-Cech compactification? (Note only the functions in $\mathcal{F}$ are guaranteed to extend continuously.)

Exercise 4.

(i) Show that a normal space is completely regular.

(ii) Show that any completely regular space is regular.

(iii) Show that the product of two completely regular spaces is completely regular.