9: CANTOR SPACES, PEANO SPACES, CONTINUA

A. CANTOR SET

Def. A space X is *totally disconnected* if the connected components of X are points. Examples: the Cantor set, the rational numbers, the irrationals.

Arbitrary products and subspaces of totally disconnected spaces are totally disconnected.

A related definition is: a space X is *zero-dimensional* if each point has a nbd basis consisting of open-closed sets: given any x and nbd U of x, there exists V open-closed so that $x \in V \subset U$; or if $A \subset X$ is closed and $x \notin A$, one may find V closed-open containing x and disjoint from A.

Clearly any zero-dimensional Hausdorff space is totally disconnected, but the converse is false.

1. A compact Hausdorff space X is totally disconnected iff whenever $x \neq y$ in X there exists an open-closed subset of X containing x but not y.

Proposition. A locally compact Hausdorff space is zero-dimensional iff it is totally disconnected.

Proof. Assuming X totally disconnected, we want to prove zero-dimensional: given $A \subset X$ closed and $x \notin A$, we want to find V clopen, containing x and disjoint from A. Let U be. precompact open neighborhood of x disjoint from A (since X is regular) , and for each $p \in \partial U$, find $V_p \subset \overline{U}$ clopen, containing x but not p (from the previous problem.) The complements (in X) V_p^c cover the compact set ∂U, so we may take a finite subcover. The intersection $V = V_{p_1} \cap \ldots \cap V_{p_n}$ is a clopen subset of $\overline{U}$, containing x and disjoint from ∂U, hence contained in U and disjoint from A.

A little Set Theory. By definition, two sets have the same cardinality if there exists a bijection between them. The *Schröder-Bernstein theorem* states: it there exist injective maps $f : A \to B$ and $g : B \to A$, then $card(A) = card(B)$.

An immediate consequence is:

$$card[0,1] = card(\mathbb{R}) = card(0,1) = card[0,1) = card(0,1].$$

(Injections are given by inclusion, and by any homeomorphism $\mathbb{R} \to (0,1)$.)

Let $\mathcal{F} = \mathcal{F}(\mathbb{N}; \{0,2\})$ be the space of all infinite sequences of zeros and twos. There is a bijection from the Cantor set to this space, given by interval labeling: each point in C is the intersection of a decreasing sequence of closed intervals (L_k) of lengths $1/3^k$, with coherent labeling: if $L_{k+1} \subset L_k$, the label of L_{k+1} is that of L_k, extended by 0 if L_{k+1} is the left third of L_k, by 2 if it is the right third of L_k.

The 'base 3 expansion' map from $\mathcal{F}$ into $[0, 1]$:

$$\omega \mapsto \sum_{i \geq 1} \frac{\omega_i}{3^i}$$

is injective, and corresponds to the inclusion of C into $[0, 1]$, for example:

$$0.0222\ldots = 2 \sum_{i=2}^{\infty} \frac{1}{3^i} = \frac{1}{3}.$$

Clearly $\mathcal{F}(\mathbb{N}; \{0, 1\})$ is in bijective correspondence with $\mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$. Thus $\mathcal{P}(\mathbb{N})$ and the Cantor set have the same cardinality. Unfortunately the map from $\mathcal{F}(\mathbb{N}; \{0, 1\})$ to $[0, 1]$ given by 'base 2 expansion':

$$\omega \mapsto \sum_{i=1}^{\infty} \frac{\omega_i}{2^i} \in [0, 1]$$

is surjective, but not injective, since of instance $0.01111\ldots$ and $0.1000\ldots$ both map to $\frac{1}{2}$. To remedy this, consider the subset $\mathcal{F}^*(\mathbb{N}; \{0, 1\})$ of sequences that are not eventually 1. This space is in bijective correspondence with $[0, 1)$ (via the formula just given), so we have the chain of bijections and inclusions:

$$\mathbb{R} \leftrightarrow [0, 1) \leftrightarrow \mathcal{F}^*(\mathbb{N}; \{0, 1\}) \hookrightarrow \mathcal{F}(\mathbb{N}; \{0, 1\}) \leftrightarrow \mathcal{F}(\mathbb{N}; \{0, 2\}) \leftrightarrow C \hookrightarrow [0, 1] \hookrightarrow \mathbb{R},$$

showing all these sets have the same cardinality (that of $\mathcal{P}(\mathbb{N})$), a perhaps surprising fact.

B. CANTOR SPACES *Def.* A metric space M is *perfect* if $M' = M$: every point of M is a cluster point of M.

1. A complete perfect metric space is uncountable. (Without completeness, $\mathbb{Q}$ is a counterexample.)

The middle-thirds Cantor set is a compact (hence complete), perfect, totally disconnected metric space. (A compact metric space is *totally disconnected* if each point has arbitrarily small clopen neighborhoods.)

Def. A metric space is a *Cantor space* if it is compact, perfect and totally disconnected.

2. Any nonempty clopen subset $A \subset M$ of a Cantor space M is a Cantor space.

Thus a Cantor space M can always be divided into two disjoint Cantor pieces. Let $p \in M$, $U \subset M$ a small clopen nbd of p. Then $M = U \sqcup U^c$.

Lemma. Given a Cantor space M and $\epsilon > 0$, one may find N so that if $d \geq N$ there exists a *partition* of M into exactly d Cantor pieces of diameter $\leq \epsilon$.