Tietze extension theorem

X normal (Hausdorff) sp $\quad C \subset X$ closed $\quad f: C \xrightarrow[\text{cont.}]{} [-1,1]$

$\Rightarrow \exists$ an extension of f to $F \in C(X; [-1,1])$

Proof $\quad E_0 = \{x \in C \mid f(x) \le -\frac{1}{3}\} \quad F_0 = \{x \in C \mid f(x) \ge \frac{1}{3}\}$

(closed, disjoint)

$\exists \quad g_0 : X \longrightarrow [-1/3, 1/3]$ cont.

$$g_0|_{E_0} \equiv -\frac{1}{3} \quad , \quad g_0|_{F_0} \equiv \frac{1}{3}$$

$$|f - g_0|(x) \le |f| + |g_0| \le \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \qquad x \in X \setminus (E_0 \cup F_0)$$

$x \in E_0 \quad -1 + \frac{1}{3} \le f - g_0 \le 0 \quad |f - g_0| \le \frac{2}{3}$

$x \in F_0 \quad 0 \le f - g_0 \le 1 - \frac{1}{3} \quad |f - g_r| \le \frac{2}{3}$

$\Big\} \; |f - g_0| \le \frac{2}{3}$ on C

$$E_1 = \{x \in C \mid f - g_0 \le -\frac{2}{9}\}, \quad F_1 = \{x \in C \mid f - g_0 \ge \frac{2}{9}\}$$

(closed, disjoint)

Let $g_1 : X \longrightarrow [-\frac{2}{9}, \frac{2}{9}]$ cont. $\begin{cases} g_1|_{E_1} \equiv -\frac{2}{9} \\ g_1|_{F_1} \equiv \frac{2}{9} \end{cases}$

on $X \setminus (E_1 \cup F_1)$

$$|f - g_0 - g_1| \le |f - g_0| + |g_1| \le \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

on E_1

$$-\frac{2}{3} + \frac{2}{9} \le f - g_0 - g_1 \le 0 \longrightarrow |f - g_0 - g_1| \le \frac{4}{9}$$

on F_1

$$0 \le f - g_0 - g_1 \le \frac{2}{3} - \frac{2}{9} = \frac{4}{9} \to |f - g_0 - g_1| \le \frac{4}{9}$$

continue

$g_0, g_1, g_2, \dots X \to [-\frac{1}{3}, \frac{1}{3}]$ cont.

$$|g_n| \le \frac{2^n}{3^{n+1}} \quad \text{on } X$$

$$\left| f - g_1 - g_2 - \cdots - g_n \right| \le \frac{2^{n+1}}{3^{n+1}} \quad \text{on } C \qquad (*)$$

consider

$$F(x) = \sum_{n=0}^{\infty} g_n(x)$$

note

$$\sum_{n=0}^{\infty} |g_n(x)| \le \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} < \infty \qquad \text{(geom. series)}$$

so by Weierstrass: $\sum g_n$ converges absolutely and uniformly on X to $F: X \to [-1, 1]$. cont.

taking limits in (*)

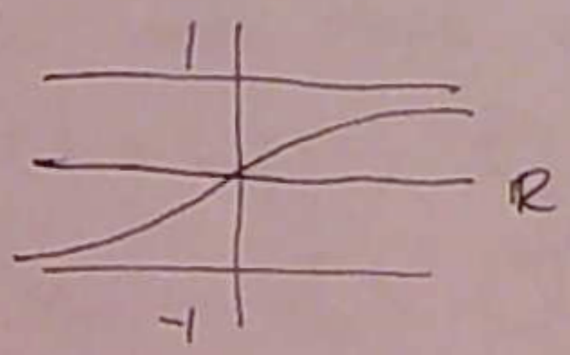
$$\left| f - \sum_{n=0}^{N} g_n \right| \le \frac{2^{N+1}}{3^{N+1}} \xrightarrow{N \to \infty} 0 \quad \text{so } f \equiv F \text{ on } C$$

($\sum_{n=0}^{N} g_n \to F$)

Any $f: C \to [-1, 1]$ can be approx by a sum of Urysohn functions (unif. over C)

Q) what if f is unbounded! $f: C \to \mathbb{R}$ cont.

$h: \mathbb{R} \to (-1, 1)$ homeo. ($C \subset X$ closed)

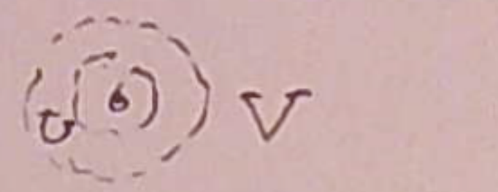


$h(t) = \frac{2}{\pi} \arctan t$.

Extend $h \circ f$ to $G: X \to [-1, 1]$ cont. extending $h \circ f$ from C to X

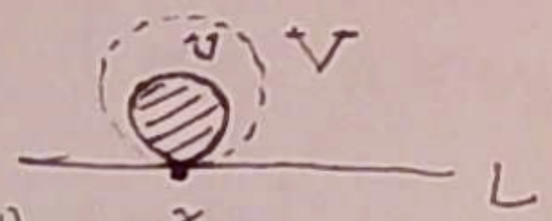
Problem What to do on the sets where $G = \pm 1$?

[12] Moore halfplane is regular H



regular $(\forall x)$ V nbd of $x \longrightarrow \exists\ U_x$ s.t. $\overline{U} \subset V$ (enough to contain V: basic set)

$x \in H \setminus L$: OK.



H not normal FIRST ATTEMPT (not sufficient)

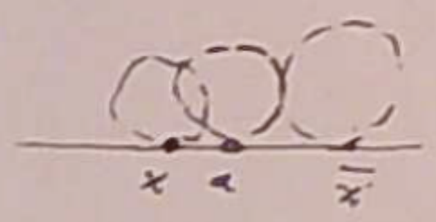
$a \in L$ $F_1 = \{a\}$ $F_2 = L \setminus \{a\}$ (both closed)

Any (basic) nbd of F_1 intersects F_2.

Any open V containg F_2 contains $W = \bigcup_{x \in L \setminus \{a\}} B_x$

Any open U containg F_1 contains B_a

claim $W \cap B_a \neq \emptyset$ [Note] not nec. true!



[16] Regular + 2nd countable $\Longrightarrow$ normal. (X Hausdorff)

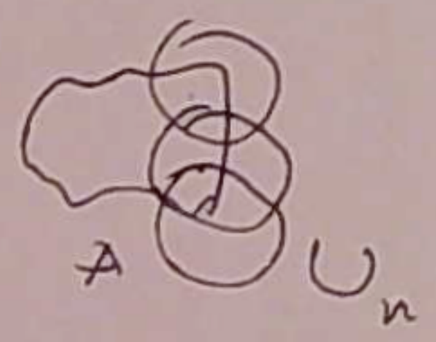
Pf $A, B \subset X$ closed $A \cap B = \emptyset$

(X regular) $x \in A$ U_x open in X $\overline{U}_x \cap B = \emptyset$ $\{U_x\}_{x \in A}$ open cover

↓ Lindelöf

$\{U_n\}_{n \geq 1}$ open cover

$B \cap \overline{U}_n = \emptyset$



$\{V_m\}_{m \geq 1}$ open cover of B $\overline{V}_m \cap A = \emptyset$

now make these covers of A (resp. B) disjoint

$\tilde{U}_1 = U_1$, $\tilde{U}_2 = U_2 \setminus \overline{V}_1$, $\tilde{U}_3 = U_3 \setminus (\overline{V}_1 \cup \overline{V}_2)$, ... $\tilde{U}_n = U_n \setminus \bigcup_{j=1}^{n-1} \overline{V}_j$ (open)

$\tilde{V}_m = V_m \setminus \bigcup_{k=1}^{m-1} \overline{V}_k$

$A \subset \bigcup_{n \geq 1} \tilde{U}_n$ (open)

$x \in A \longrightarrow x \in U_{n_0}$ and in no $\overline{V}_j$, so is in $\tilde{U}_{n_0}$

Also $B \subset \bigcup_{m \geq 1} \tilde{V}_m$ (open)

Claim $U = \bigcup_{n \geq 1} \tilde{U}_n$, $V = \bigcup_{m \geq 1} \tilde{V}_m$ are disjoint.

if $x \in U \cap V$ $\quad x \in \tilde{U}_n \cap \tilde{V}_m$ (say $n \leq m$)

$x \in \tilde{U}_2 = U_1 - \overline{V}_1$ $\quad (x \in U_1)$

$x \in \tilde{V}_4 = V_4 - (\overline{U}_1 \cup \overline{U}_2 \cup \overline{U}_3)$ $\quad (x \notin \overline{U}_1)$

> contr

The Moore plane is not normal

notation $H = \{(x,y) ; y \geq 0\} \subset \mathbb{R}^2$ $\quad H = U \cup L$ ($U = \{y>0\}$, $L = \{(x,0) ; x \in \mathbb{R}\}$)

For $x \in L$, $B_x(r)$ is the open disk in U (radius $r>0$) tangent to L at x, union $\{x\}$

$\mathbb{Q} \subset L$ rationals $\quad \mathbb{I} \subset L$ irrationals $\quad \mathbb{Q} \cup \mathbb{I} = L$ (disjoint union)

Since (H, Moore) induces in L the discrete topology, $\mathbb{Q}$ and $\mathbb{I}$ are disjoint and closed in (H, Moore)

Claim 1 If U, V are open in (H, Moore), $U \supset \mathbb{Q}$ and $V \supset \mathbb{I}$, then U and V cannot be disjoint.

Note U contains an open set of the form $\bigcup_{x \in \mathbb{Q}} B_x(r_x)$

V contains an open set of the form $\bigcup_{y \in \mathbb{I}} B_y(s_y)$

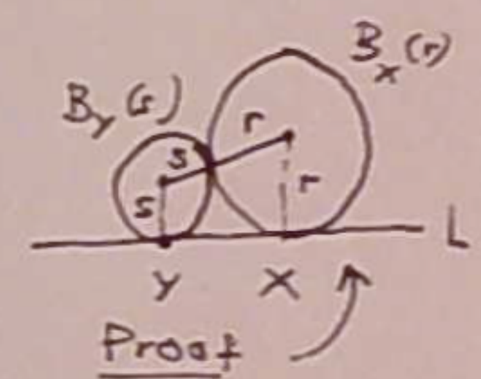
for two functions $x \mapsto r_x$ (from $\mathbb{Q}$ to $\mathbb{R}_+$), $y \mapsto s_y$ (from $\mathbb{I}$ to $\mathbb{R}_+$).

(not nec. continuous)

If $U \cap V = \emptyset$, then $B_x(r_x) \cap B_y(s_y) = \emptyset \quad \forall x \in \mathbb{Q}, \forall y \in \mathbb{I}$.

Geometric Lemma For $x, y \in L$ and $r, s > 0$:

$$B_x(r) \cap B_y(s) = \emptyset \Rightarrow |x-y| \geq 2\sqrt{r \cdot s}$$



Proof

For each $n \geq 1$, consider the set

$$R_n = \{y \in \mathbb{I} \mid s_y \geq \tfrac{1}{n}\}$$

note $\mathbb{Q} \cup \left(\bigcup_{n \geq 1} R_n\right) = \mathbb{R}$.

Claim 2 The R_n cannot all be nowhere dense* in $\mathbb{R}$ (w/ the usual metric topology on $\mathbb{R}$)

* next page

Pf (claim 2)

By the Baire property of $\mathbb{R}$, a countable union of nowhere dense sets has empty interior in $\mathbb{R}$ (so it can't be all of $\mathbb{R}$). ▨

(of course the one-point sets $\{t\}$, $t \in \mathbb{Q}$ are nowhere dense)

Proof of claim 1

Thus some R_{n_0} is not "nowhere dense": its closure (in $(\mathbb{R}, \text{metric})$) contains an open interval (a, b). In particular we may find

$$x_0 \in (a,b) \cap \mathbb{Q} \text{ and } y_j \in R_{n_0} \text{ s.t. } y_j \longrightarrow x_0 \text{ (in the metric topology).}$$

But by assumption $B_{x_0}(r_{x_0}) \cap B_{y_j}(s_{y_j}) = \emptyset$, and $s_{y_j} \geq \frac{1}{n_0}$.

Now the geometric lemma implies:

$$|x_0 - y_j| \geq 2\sqrt{r_{x_0} \cdot s_{y_j}} \geq \frac{2\sqrt{r_{x_0}}}{\sqrt{n_0}}.$$

this contradicts $y_j \longrightarrow x_0$.

⊛ X: Hausdorff top. space

Def. A set $M \subset X$ is "nowhere dense" (or "meager") if its closure $\overline{M}$ has empty interior.

X has the Baire property if a countable union of nowhere dense subsets of X has empty interior

Theorem If X is either

(i) a complete metric space; or

(ii) a locally compact Hausdorff space.

Then X has the Baire property.

Ref. [Munkres § 48; Gamelin-Greene § p.11].