PROBLEM SET 3: BAIRE SPACES, G_δ SETS

Definition. X is a *Baire space* if a countable intersection of open, dense subsets of X is dense in X.

1. Complete metric spaces are Baire spaces.

2. *Def:* A set $E \subset X$ is *nowhere dense* in X if its closure $\overline{E}$ has empty interior.

Show: X is a Baire space $\Leftrightarrow$ any countable union of nowhere dense sets has empty interior.

3. *Def.* X is *locally compact* if for all $x \in X$, and all open U_x, there exists V_x with compact closure $\overline{V}_x \subset U_x$.

Show: X is locally compact $\Leftrightarrow$ for all $C \subset X$ compact, and all open $U \supset C$, there exists V open with compact closure, so that: $C \subset V \subset \overline{V} \subset U$.

4. Locally compact Hausdorff spaces are Baire spaces.

Problems 5, 5.5, 6, 7 are a *review of compact spaces.*

5. (i) X Hausdorff, $K \subset X$ compact $\Rightarrow K$ is closed in X.

(ii) X compact Hausdorff, $C \subset X$ closed $\Rightarrow C$ compact.

(iii) X, Y Hausdorff, X compact, $f : X \to Y$ continuous, injective $\Rightarrow f$ is an embedding.

5.5 $C \subset X$ compact subset, $F : X \to Y$ continuous $\Rightarrow f(C)$ is a compact subset of Y.

6. (i) X is compact iff any family $\{C_\lambda\}_{\lambda \in A}$ of closed subsets of X with the *finite intersection property* has non-empty intersection:

$$\Big(\bigcap_{\lambda \in F} C_\lambda \neq \emptyset \quad \forall F \subset A \text{ finite }\Big) \Rightarrow, \bigcap_{\lambda \in A} C_\lambda \neq \emptyset.$$

(ii) $K_1 \supset K_2 \supset \ldots$, K_n compact nonempty $\Rightarrow \bigcap_{n \geq 1} K_n \neq \emptyset$.

7. Compact Hausdorff spaces are normal.

8. Locally compact Hausdorff spaces are completely regular.

Hint. Let $F \subset X$ closed, $p \notin F$. Use problem 3 to find open, precompact neighborhoods U_1, U_2 of p so that:

$$p \in U_1 \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset F^c.$$

Using problem 7, let $f \in C(\overline{U_2}, I)$ be a Urysohn function for $(p, \overline{U_2} \setminus U_1)$, $f(p) = 1$. Let $F : X \to I$ be the extension of f by zero on $(\overline{U_2})^c$. Use the "pasting lemma" ([Munkres, p.108]) to show F is continuous.

9. A complete metric space without isolated points is uncountable. (*Hint:* Baire property, complements of one-point sets.)

10. *Uniform boundedness principle.* X complete metric, $\mathcal{F} \subset C(X)$ a family of continuous functions, bounded at each point:

$$(\forall a \in X)(\exists M(a) > 0)(\forall f \in \mathcal{F})|f(a)| \leq M(a).$$

Then there exists a nonempty open set $U \subset X$ so that $\mathcal{F}$ is eqüibounded over U–there exists a constant $C > 0$ so that:

$$(\forall f \in \mathcal{F})(\forall x \in U)|f(x)| \leq C.$$

Hint: For $n \geq 1$, consider $A_n = \{x \in X; |f(x)| \leq n, \forall f \in \mathcal{F}\}$. Use Baire's property.

11. *Uniform boundedness for linear operators.*

Let E, F be Banach spaces. Consider a family $\mathcal{F} \subset \mathcal{L}(E; F)$ of bounded linear operators from E to F. Suppose $\mathcal{F}$ is bounded on each vector:

$$(\forall v \in E)(\exists M(v) > 0)(\forall T \in \mathcal{F})||Tv|| \leq M(v).$$

Then the operators in $\mathcal{F}$ are uniformly bounded:

$$(\exists C > 0)(\forall T \in \mathcal{F})||T|| \leq C.$$

(Here $||T|| = \sup\{||Tv||; v \in E, ||v|| \leq 1\} < \infty$.)

Problems 12 to 19 deal with G_δ sets.

12. Any closed subset of a metric space is a G_δ (use the distance function.)

13. Let X be normal $A \subset X$ non-empty. There exists $f \in C(X; [0,1])$ with $\{x \in X; f(x) = 0\} = A$ iff A is a closed G_δ subset of X.

Hint. One direction is easy. For the other, suppose $A = \bigcap_{n \geq 1} U_n$, where we may assume (show this) $U_n \supset U_{n+1}$. Let $f_n \in C(X; [0,1])$ be a solution to the Urysohn problem for the pair of closed sets (A, U_n^c), with $f_n \equiv 0$ in A, $f_n \equiv 1$ in U_n^c. Now consider:

$$f : X \to [0,1], \quad f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x).$$

Show that f is continuous with values in $[0,1]$, and that $x \notin A \Rightarrow f(x) > 0$. Note that if $B \subset X$ is closed and disjoint from A, f can be chosen to satisfy $f \equiv 1$ in B (just assume $B \subset U_n^c$ for all n.)

14. *(Strong Urysohn Lemma.)* Let X be normal, $A, B \subset X$ closed disjoint. Show there exists $f \in C(X; [0,1])$ so that $f^{-1}(0) = A, f^{-1}(1) = B$ iff A and B are G_δ sets.

Hint: If A, B are closed G_δ sets, let $f_A, f_B \in C(X, [0,1])$ be as in the previous problem:

$$f_A^{-1}(0) = 1, \quad f_A \equiv 0 \text{ in } B; \quad f_B^{-1}(0) = B, \quad f_B \equiv 0 \text{ in } A.$$

Then consider:

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}.$$

15. If $f : X \to Y$ is any function (X topological space, Y metric space), the set of continuity $C_f \subset X$ is a G_δ set.

Hint: Let U_n be the union of all open sets V such that the diameter of $f((V)$ is less than $1/n$. Show $C_f = \bigcap_{n \geq 1} U_n$.

16. (i) $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$ (hence there are no functions $f : \mathbb{R} \to \mathbb{R}$ with continuity set $C_f = \mathbb{Q}$). (See the handout.)

(ii) The irrationals $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ are a G_δ set.

(iii) In a separable Baire space without isolated points, no countable dense subset is a G_δ.

Theorem. X: Baire space, Y: metric space, $f_n : X \to Y$ continuous, $f_n \to f$ pointwise. Then C_f is a dense G_δ set in X.([Munkres, p. 297]).

17. Let $f_n : \mathbb{R} \to \mathbb{R}$ continuous. Suppose $f_n \to f$ pointwise on $\mathbb{R}$. Then f is continuous at uncountably many points of $\mathbb{R}$.

18. X: complete separable metric space, $D \subset X$ countable dense set. Show there does not exist a function $f : X \to \mathbb{R}_+$ (positive reals), continuous or not, satisfying:

$$d(x, y) \geq f(x)f(y), \quad \forall x \in D, \forall y \in X \setminus D.$$

19. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that the derivative $g = f' : \mathbb{R} \to \mathbb{R}$ is the pointwise limit of a sequence of continuous functions. (As a consequence, the set of continuity of g is a dense G_δ set.)

Remark: The converse is true: for any dense G_δ set $A \subset \mathbb{R}$, there exists $f : \mathbb{R} \to \mathbb{R}$ differentiable, so that the continuity set of f' is A.(*BAMS*, 4/2019.)

20. In a second countable space, any family $\{U_\lambda\}$ of nonempty disjoint open sets is necessarily countable. (*Hint:* The subspace $S = \bigcup_\lambda U_\lambda$ is second countable, and has an obvious open cover. Use Lindelöf's theorem, and consider if this cover has any subcovers.)