PROBLEM SET 7: FUNCTION SPACES, STONE-WEIERSTRASS

1. [Munkres p. 289] X locally compact, σ-compact; (Y, d) metric. Then the u.o.c. topology on $\mathcal{F}(X, Y)$ is metrizable, with a complete metric if d is complete.

Hint: Let (K_i) be a compact exhaustion of X, $K_i \subset int(K_{i+1})$. Letting $\mathcal{F}_i = \mathcal{F}(K_i, Y)$ with the uniform topology, show $\mathcal{F}(X, Y)$ (u.o.c.) is homeomorphic to a closed subset of the product of the $\mathcal{F}_i$.

2. [Munkres, p. 288] The compact-open topology in $C(X, Y)$ is Hausdorff if Y is Hausdorff, regular if Y is regular.

3. [Munkres, p.292, problem 1] (4 items).

4. [cp. Munkres, p. 293]: X is locally compact, σ-compact. $f_n : X \to \mathbb{R}^k$. If (f_n) is equicontinuous on compact sets and bounded at each point, there exists a subsequence converging u.o.c. to a continuous function. (You may use the general A-A theorem in the notes.) What if $\mathbb{R}^k$ is replaced by a general proper metric space (Y, d)?

5. Let $X = [0, \infty)$. Show that for each continuous $f : X \to \mathbb{R}$ there exists a sequence of the form:

$$p_k(x) = \sum_{n=0}^{n_k} a_n e^{-nx}$$

such that $p_k \to f$ uniformly on compact sets.

Hint: Verify the hypotheses of Stone-Weierstrass for the function algebra on X generated by 1 and e^{-x}.

6. Trigonometric polynomials:

$$p(x) = \sum_{k=0}^{N} (a_k \cos kx + b_k \sin kx)$$

are dense in the space of 2π-periodic continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the topology of uniform convergence.

7. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^1 x^n f(x)dx = 0$ for each $n = 0, 1, 2, ...,$ then $f(x) \equiv 0$ on $[0, 1]$. *Hint:* show $\int_0^1 f^2(t)dt = 0$.

7.5 If $f \in C(\mathbb{R}^n)$, there exists a sequence p_j of polynomials in n variables so that $p_j \to f$ u.o.c. in $\mathbb{R}^n$. If $f(0) = 0$, we may require the approximating polynomials to satisfy $p_j(0) = 0$ for all j.

7.7 The set of continuous, piecewise linear functions is dense in $C(\mathbb{R})$ (for the u.o.c. topology).

8. The space $\mathcal{F}_p(\mathbb{R}, \mathbb{R})$ of all functions (with the topology of pointwise convergence) is not compactly generated.

Hint: Let T be the subset consisting of all f such that $f(x) = n$ for all but at most n values of x (where $n \in \mathbb{N}$), $f(x) = 0$ otherwise. Then $T \cap K$ is closed in K for each compact subset K of this product space, but T is not closed.

9. Let X be locally compact Hausdorff. If X is σ-compact with compact exhaustion $(K_n)_{n\geq 1}$, define a metric on $C(X)$ by:

$$\rho(f,g) = \sum_{n=1}^{\infty} \rho_n(f,g),$$

$$\rho_n(f,g) = \min\{\frac{1}{2^n}, \sup_{x \in K_n} |f(x) - g(x)|\}.$$

Show that ρ metrizes the u.o.c. topology on $C(X)$.

10. Let X be any space, (Y, d) metric. For each $f : X \to Y, \epsilon > 0, C \subset X$ compact, consider the subset of $\mathcal{F}(X, Y)$:

$$B_C(f, \epsilon) = \{g : X \to Y; d(f(x), g(x)) < \epsilon, \forall x \in C\}.$$

Show these sets form the basis of a topology in $\mathcal{F}(X, Y)$.

11. [Munkres p. 288, no. 5].

12. (i) Exhibit a countable dense subset of $\mathcal{F}(I, I)$ with the pointwise topology. ($I = [0, 1]$.)

(ii) Is $\mathcal{F}(I, I)$ separable with the topology of uniform convergence?

13. X: metric space, E: Banach space. $f : X \to E$ is *compact* if $A \subset X$ bounded $\Rightarrow \overline{f(A)}$ is compact. f is *finite-dimensional* if $f(X)$ is contained in a finite-dimensional subspace of E.

(i) A u.o.c. limit of compact maps is compact.

(ii) f is compact iff it is the u.o.c. limit of finite-dimensional maps.

14. Prove that the subspace $\mathbb{P} \subset \mathbb{R}$ of irrational numbers is homeomorphic to $\mathcal{F}_p(\mathbb{N}, \mathbb{N})$ (topology of pointwise convergence.) *Hint:* continued fraction expansions.

15. If $f_n \to f$ pointwise in X and $\mathcal{F} = \{f_1, f_2, \ldots\} \subset C(X)$ is equicontinuous, then $f \in C(X)$ and $f_n \to f$ u.o.c.

16. (Dini) If $f_1 \leq f_2 \leq f_3 \leq \ldots$ is an increasing sequence of functions in $C(X)$ converging pointwise to $f \in C(X)$, then$f_n \to f$ u.o.c. (*Hint:* ETS the set $\{f_1, f_2, \ldots\}$ is equicontinuous.)

17. On the set of homeomorphisms of the real line, the pointwise topology and the u.o.c. topology coincide.

18. If X is any space and M is complete metric, let $\mathcal{F} = \{f_n\}_{n\geq 1} \subset C(X; M)$ be a countable equicontinuous set. If $f_n(x)$ converges for all x in a dense subset $D \subset X$, show that f_n converges u.o.c in X.

Supplementary problems on compactness.

19. (i) If $Y_1, Y_2, \ldots$ are sequentially compact, then $Y = \Pi_{n\geq 1} Y_n$ is sequentially compact.

(ii) If N is countable and Y is sequentially compact, $\mathcal{F}_p(N, Y)$ is sequentially compact.

20. (Application of 19–Helly's theorem). Let $X \subset \mathbb{R}$ be arbitrary, $f_n : X \to [a, b]$ a sequence of monotone functions (say nondecreasing.) Then (f_n) has a convergent subsequence (pointwise in X). *Hint:* Show f_n has a subsequence converging pointwise in a countable dense subset of X, then use monotonicity.

21. Let Y be compact Hausdorff, X arbitrary. Then $\pi : X \times Y \to X$ is a closed map. (*Hint:* tube lemma). This is false if Y is not compact.

22. Let $f : X \to Y$ be a map (X a space, Y compact Hausdorff). (i) If the graph $\Gamma_f \subset X \times Y$ is closed, f is continuous. (Hint: if $V \subset Y$ is a nbd of $f(x_0)$, $C = \Gamma_f \cap (X \times V^c)$ is closed; consider its image under π, the standard projection from $X \times Y$ to X.) This is false if Y is not compact.

(ii) If f is continuous, Γ_f is closed (Y compact not needed, just Hausdorff.)

23. Prove the *weak Tychonoff theorem:* If M_i are compact metric spaces, then the product $\Pi_{i\geq 1} M_i$ is a compact metric space.

24. Use the weak Tychonoff theorem to state and prove a sequential Arzela-Ascoli theorem for subsets $\mathcal{F} \subset C(X; M)$, where X is locally compact second countable and M is metric.