COMPACTIFICATIONS AND STONE-CECH

Def. A *compactification* of a Hausdorff space X is a pair $(\hat{X}, \varphi)$, where $\hat{X}$ is compact Hausdorff and $\varphi : X \to \hat{X}$ is an embedding (homeomorphism onto $\varphi(X) \subset \hat{X}$), with $\varphi(X)$ dense in $\hat{X}$.

An example is the Alexandroff compactification of a locally compact space X, which adds a single 'point at infinity' to X to obtain $\hat{X}$. The continuous bounded functions on X that extend to the Alexandroff compactification are exactly those tending to a constant 'at infinity'.

Recall the following general construction: for a family $\mathcal{F} \subset C_b(X)$ (the space of bounded continuous functions from X to $\mathbb{R}$), consider the product of closed intervals indexed by $\mathcal{F}$:

$$P = \Pi_{f \in \mathcal{F}} I_f, \qquad I_f = [\inf f, \sup f] \subset \mathbb{R}$$

and the 'evaluation map':

$$e : X \to P, \qquad [e(x)]_f = f(x) \in I_f.$$

P is compact Hausdorff, by Tychonoff's theorem. We know:

(i) e is continuous; (ii) if $\mathcal{F}$ separates points, e is injective; (iii) if $\mathcal{F}$ separates points from closed sets, e is an open map onto $\varphi(X)$, with the induced topology. Hence $\hat{X} = \overline{\varphi(X)}$ (closure in P) is a compactification of X, associated with $\mathcal{F}$.

(Informally, we may think of X as a subset of $\hat{X}$, identifying X with $\varphi(X)$.)

Let $\pi_f : P \to I_f$ be the standard projection. Then $\hat{f} = (\pi_f)_{|\hat{X}} \in C(\hat{X}; I_f)$ satisfies $\hat{f} \circ e = f$, and under the identification just mentioned $\hat{f}$ extends f to $\hat{X}$ (any $f \in \mathcal{F}$ can be so extended.)

1. If X is locally compact and $\mathcal{F}$ is the family of continuous real-valued functions on X tending to a constant at infinity, show (i) $\mathcal{F}$ separates points from closed sets; (ii) the compactification associated to $\mathcal{F}$ is homeomorphic to Alexandroff's.

If X is (Hausdorff and) completely regular, then $\mathcal{F} = C_b(X)$ separates points from closed sets. The associated compactification is called the *Stone-Cech compactification* of X, traditionally denoted βX. As just noted, any $f \in C_b(X)$ extends continuously to βX (under identification via the evaluation embedding $e : X \to P$). A more general 'universal extension property' is true:

Theorem. If X is completely regular (Hausdorff), K is compact Hausdorff and $h : X \to K$ is continuous, there exists $\bar{h} : \beta X \to K$ extending h, in the sense that $\bar{h} \circ e = h$.

Uniqueness. Let X be completely regular (Hausdorff), (Y_1, e_1) and (Y_2, e_2) two compactifications of X with the universal extension property. Then they are equivalent: homeomorphic via $f_2 : Y_1 \to Y_2$ satisfying $f_2 \circ e_1 = e_2$, with inverse $f_1 : Y_2 \to Y_1$ satisfying $f_1 \circ e_2 = e_1$.

2. *Lemma* Let X, Y be Hausdorff, $A \subset X$, $f : A \to Y$ continuous. Then there exists at most one continuous extension of f to a continuous map from $\overline{A}$ to Y.

3. Using the lemma, prove the uniqueness statement. *Hint:* Find f_2 as the extension of e_2 to Y_2 (meaning $f_2 \circ e_1 = e_2$). Show that $f_2 \circ f_1$ is the identity on the dense subset $e_2(X)$ of Y_2.

Problems 4,5,6 are from [Munkres].

4. If (Y, e_Y) is any compactification of X ($e_Y : X \to Y$ the embedding), there exists $F : \beta X \to Y$ continuous surjective closed map, satisfying $F \circ e_\beta = e_y$. (In this sense, βX is the 'maximal' compactification of X.)

5. X is connected iff βX is connected.

6. (i) If X is normal and $y \in \beta X \setminus X$, then y is *not* the limit of a sequence of points in X.

(ii) If X is completely regular (Hausdorff) and noncompact, then βX is not metrizable.

Remark: In particular if X is locally compact metric, non-compact, then βX is not metrizable (and is separable, and connected, if X is.) This gives lots of examples of compact, connected, separable non-metrizable spaces.

7. If $\mathcal{F}_1 \subset \mathcal{F}_2$ are two families in $C_b(X)$, both separating points from closed sets, with associated compactifications $((Y_1, e_1), (Y_2, e_2)$, there exists a continuous map $F : Y_2 \to Y_1$ satisfying $F \circ e_2 = e_1$.

Question. If $\mathcal{F}_1$ is dense in $\mathcal{F}_2$, does it follow that Y_1 and Y_2 are homeomorphic?

8. If $\mathcal{F} \subset C_b(X)$ is a countable family separating points from closed sets, the associated compactification $\hat{X}$ of X is second countable, hence metrizable. (Prove first that the product of countably many second countable spaces is second countable.)

9. If X is normal (Hausdorff), noncompact and second countable, there exists a countable family $\mathcal{F} \subset C_b(X)$ separating points from closed sets. Use this to prove the Urysohn metrization theorem. The associated compactification is second countable and metrizable. (But the Stone-Cech is not, see problem 6.)

10. Prove that if X is locally compact, σ-compact, noncompact (Hausdorff), then $C_b(X)$ is separable metric (with the uniform topology). Thus there exists a countable family $\mathcal{F} \subset C_b(X)$, dense in $C_b(X)$. Prove that $\mathcal{F}$ separates points from closed sets. How does the associated compactification (which is metrizable) relate to Stone-Cech?

A rigidity property of compact Hausdorff topologies.

Proposition. Let $\tau_1 \subset \tau_2$ be two topologies on a set X (so τ_2 is finer than τ_1. If τ_1 is Hausdorff and τ_2 is compact, then $\tau_1 = \tau_2$.

Informally if τ is a compact Hausdorff topology on X, you can't make it coarser without losing 'Hausdorff', or finer without losing 'compact'.

Recall the following:
(i) compact subsets of Hausdorff spaces are closed;
(ii) closed subsets of compact spaces are compact (from the definition–Hausdorff not needed,)

Proof of prop. Let $F \subset X$ be τ_2-closed, hence τ_2 compact (by (ii)). Since $\tau_1 \subset \tau_2$, F is τ_1-compact, hence (by (i)) τ_1-closed. Hence $\tau_2 \subset \tau_1$, so $\tau_1 = \tau_2$.

11. Let $\mathcal{F}$ is a family of mappings $f : X \to Y_f$ (X:set, Y_f: Hausdorff space depending on f.) If $\mathcal{F}$ separates points, the $\mathcal{F}$-topology of X is Hausdorff. (Recall this the weakest topology on X making all f continuous. $\mathcal{B} = \{f^{-1}(U_f); U_f \text{ open in } Y_f\}$ is a basis.

12. Let X be compact. Suppose $\mathcal{F} \subset C(X; R)$ is a countable family of real-valued functions on X. Assume $\mathcal{F}$ separates points. Then X is metrizable.

Hint: Let $\mathcal{F} = (f_n)_{n \geq 1}$, where we may assume $|f_n| \leq 1$ on X. Consider the function on $X \times X$:

$$d(p,q) = \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(q)|.$$

Let τ be the topology on X. Explain why d is $\tau \times \tau$-continous on $X \times X$, so the sets $B_r(p) = \{q \in X | d(p,q) < r\}$ are τ-open. Let τ_d be the metric topology defined by d. Then $\tau_d \subset \tau$, so they must ee equal.