MATH 561–TOPOLOGY 1–MIDTERM–October 14, 2020.

Closed books, closed notes, no asking the internet. Time given: 55 min. All spaces assumed Hausdorff, except in problem 4. Include as much detail in your answers as time allows.

1. Let M be a locally compact, σ-compact, non-compact metric space.

(i) Show that M has a countable basis.

(ii) Show that the Alexandroff compactification $M^* = M \sqcup \{\omega\}$ is metrizable.

2. Let X be a regular space. Prove that any two distinct points $x \neq y$ in X admit open neighborhoods with disjoint closures.

3. (i) Prove: The product of countably many second countable spaces is second countable.

(ii) Let $\mathcal{F} \subset C_b(X)$ be a countable family of bounded continuous functions on.a space X, separating points from closed sets. Describe the construction of the compactification $(\hat{X}, e)$ of X associated to $\mathcal{F}$. (Where e is the embedding.)

(iii) Explain why $\hat{X}$ is metrizable.

4. (i) Let $f : X \to Y$ be a continuous, surjective, closed map. If $U \subset X$ is open, there exists $V \subset Y$ open so that $f(U) \supset V$. *Hint:* consider $f(U^c)^c$.

(ii) Assume, in addition, that the 'fibers' $f^{-1}(y)$ of f are compact, for all $y \in Y$. Prove that if X is Hausdorff, then so is Y. (Prove first that disjoint compact subsets of a Hausdorff space have disjoint open neighborhoods.)