PART I: TOPOLOGICAL SPACES

DEFINITIONS AND BASIC RESULTS

1.Topological space/ open sets, closed sets, interior and closure/ basis of a topology, subbasis/ 1st ad 2nd countable spaces/ Limits of sequences, Hausdorff property.

Prop: (i) condition on a family of subsets so it defines the basis of a topology on X. (ii) Condition on a family of open subsets to define a basis of a given topology.

2. Continuous map between spaces (equivalent definitions)/ finer topologies and continuity/homeomorphisms/metric and metrizable spaces/ equivalent metrics vs. quasi-isometry.

3. Topological subspace/ relatively open (or closed) subsets. Product topology: finitely many factors, arbitrarily many factors.

Reference: Munkres, Ch. 2 sect. 12 to 21 (skip 14) and Ch. 4, sect. 30.

PROBLEMS

PART (A)

1. Define a non-Hausdorff topology on $\mathbb{R}$ (other than the trivial topology)

1.5 Is the 'finite complement topology' on $\mathbb{R}$ Hausdorff? Is this topology finer or coarser than the usual one? What does $\lim x_n = a$ mean in this topology? (*Hint:* if $b \neq a$, there is no constant subsequence equal to b.)

2. Let $Y \subset X$ have the induced topology. $C \subset Y$ is closed in Y iff $C = A \cap Y$, for some $A \subset X$ closed in X.

2.5 Let $E \subset Y \subset X$, where X is a topological space and Y has the induced topology. Thenn $\overline{E}^Y$ (the closure of E in the induced topology on Y) equals $\overline{E} \cap Y$, the intersection of the closure of E in X with the subset Y.

3. Let $E \subset X$, E' be the set of cluster points of E. Then $\bar{E} = E \cup E'$.

4. If X is first countable and $a \in E'$, then one may find a sequence $(x_n)_{n\geq 1}$ in E, so that $\lim x_n = a$.

5. (*Sorgenfrey line*, denoted $\mathbb{R}_l$) (i) The collection of subsets of $\mathbb{R}$ $\mathcal{B} = \{[a,b); a < b\}$ (left-closed intervals) define the basis of a topology on $\mathbb{R}$.

(ii) This topology is Hausdorff, and is finer than the usual topology on $\mathbb{R}$.

(iii) $\lim x_n = a$ in $\mathbb{R}_l$ iff $x_n \to a_+$ in the usual topology (one-sided limit from the right.)

6. Let $\mathcal{S}$ be a subbasis for a topology on Y. $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open $\forall U \in \mathcal{S}$.

7. (i) In a product space $X_1 \times X_2$, the canonical projections p_1, p_2 are open maps.

(ii) $f : Y \to X_1 \times X_2$ is continuous iff $p_1 \circ f$ and $p_2 \circ f$ are.

(Q) How about arbitrary products?

8. First countable, second countable, Hausdorff are preserved under finite products: $X = X_1 \times X_2$ has those properties, if each X_i does. (Q) How about arbitrary products?

8.5. *Examples:* The Moore half-plane and the non-tangential half-plane.

Show that $(H, circle)$ is coarser than (H, NT), and that any sequence in (H, NT) converging to a boundary point converges also in $(H, circle)$; but not conversely. (*Later:* (H, NT) is normal, unlike $(H, circle)$.)

8.7 The boundary line of $(H, circle)$ (i) inherits the discrete topology as a subspace; (ii) is a closed subset of $(H, circle)$; (iii) shows that a subspace of a separable space need not be separable (in contrast to the properties 1st countable or second countable.)

PART (B)

9. Any metrizable space is Hausdorff and first countable.

10. In a metric (or metrizable) space, $E \subset X$ is closed iff E is sequentially closed.

11. (X, d), $E \subset X, \neq X$, $d(x, E) := \inf\{d(x, y); y \in E)\}$. (i) $f(x) = d(x, E)$ is continuous on X (and Lipschitz)

(ii) $d(x, E) = d(x, \bar{E})$.

12 (X, d) and $(X, \min\{d, 1\})$ are equivalent metric spaces (i.e., the identity map is a homeomorphism.)

13. (i) Two quasi-isometric metrics on X define the same topology (i.e., are equivalent.)

(ii) $d_1(x, y) = |x - y|$ and $d_2(x, y) = |x^3 - y^3|$ define equivalent metrics oon $\mathbb{R}$ which are not quasi-isometric.

14. $\mathbb{R}^n$ is second countable.

15. X (topological space) second countable $\Rightarrow$ X first countable and separable.

16.* X (topological space) second countable $\Rightarrow$ any open cover admits a countable subcover (Lindelöf)

17. X metrizable and separable $\Rightarrow$ X second-countable.

18. The Sorgenfrey line is first countable and separable, but not second countable (and hence is not metrizable.)

PART (C)

19. Let $X = \mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$, the space of *all* functions from $\mathbb{R}$ to $\mathbb{R}$, with the product topology.

(i) Describe explicitly a basis for this topology (and verify that it satisfies the conditions for a basis);

(ii) Show that $\lim f_n = f$ in this topology iff $f_n(t) \to f(t)$, $\forall t \in \mathbb{R}$ (pointwiise).

20.* Show that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is not first countable (hence not metrizable).

21.* Let $E \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of characteristic functions of finite sets. The constant function 1 is in E', but there is no sequence $f_n \in \mathcal{F}$ so that $\lim f_n = f$.

22. A separable metric space cannot contain an uncountable discrete set.

23. $C(\mathbb{R}; [0, 1])$ (with the uniform metric) is a metric space without a countable basis (equivalently, not separable.)

Hint: For $S \subset \mathbb{Z}$, define $f(n) = 1$ if $n \in S$, $f(n) = 0$ if $n \in \mathbb{Z} \setminus S$, and continuous and linear otherwise. Then the set $\{f_S; S \subset \mathbb{Z}\}$ is uncountable and discrete: $d(f_S, f_T) = 1$ if $S \neq T$.

24. Let X be an infinite set, (M, d) a metric space with at least two elements. Then $\mathcal{B}(X; M)$ (the space of bounded functions from X to M, with the uniform metric) does not have a countable basis.