2. EXISTENCE AND EXTENSION OF CONTINUOUS FUNCTIONS

1. M, N metric spaces, $f : M \to N$ continuous, $A \subset M$. Suppose for all $a \in \overline{A}$, the limit $\lim_{x\to a} f(x) := F(a)$ exists. Then the extension of f to $F : \overline{A} \to N$ is continuous.

For general topological spaces, the corresponding result needs a new condition.

Definition. A Hausdorff space X is *regular* if

$$(\forall x \in X)(\forall U_x)(\exists V_x \subset U_x \text{ open })(\overline{V_x} \subset U_x).$$

Remark: This condition is local: for each x, existence of a 'basis of closed neighborhoods' of x.

2. Equivalently, X is regular iff Hausdorff and for all $x \in X$ and all $C \subset X$ closed with $x \notin C$, we may find U_x and $W \supset C$ open, so that $U_x \cap W = \emptyset$.

3. Metric spaces are regular.

4. $(H, halfdisk)$, the upper half plane with the half-disk topology, is Hausdorff but not regular.

Hint: Let D_x be a half-disk subbasic neighborhood ($x \in L$), C its complement (a closed set disjoint from x). Then the closure $\overline{V}$ of any neighborhood V of x intersects C, so $\overline{V}$ cannot be contained in the complement of C.

5. Let X, Y be Hausdorff topological spaces with Y regular. Assume $A \subset X$, $f : A \to Y$ is continuous and $\lim_{x\to a} f(x) := F(a)$ exists, for each $a \in \overline{A}$. Then the extension of f defined by this limit is a continuous map $F : \overline{A} \to Y$.

6. If X, Y are metric spaces, Y is complete, $A \subset X$ and $f : A \to Y$ is uniformly continuous on A, then there is a unique extension of f to a continuous map $F : \overline{A} \to Y$.

Hint: Existence has two parts: defining the map and proving it is continuous. Uniqueness is easy.

7. The continuous surjective image of a separable space is separable. That is, if $f : X \to Y$ is continuous and onto, and X is separable, then so is Y.

8. The space of Lipschitz functions $f : [0, 1] \to \mathbb{R}$, with the topology defined by the norm:

$$||f|| = |f(0)| + [f], \quad [f] = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

is not separable.

9. (i) A discrete uncountable space cannot be second-countable.

(ii) The Moore plane is separable and first-countable, but not second-countable (hence not metrizable.)

Hint: A subspace of a second-countable space is second-countable; then use part (i).

10. In the linear space $X = C[0,1]$ of continuous real-valued functions in [0,1], consider the supremum and L^1 norms:

$$||f||_{sup} = \sup\{|f(t)|; t \in [0,1]\}; \quad ||f||_1 = \int_0^1 |f(t)| dt.$$

(i) Any L^1 ball contains a sup ball:

$$B_{L^1}(f_0, R) \supset B_{sup}(f_0, R), \forall f_0 \in X, R > 0.$$

(ii) The L^1 ball $B_{L^1}(0,1)$ is not contained in any ball $B_{sup}(0, R)$. Thus these two norms define different topologies on X. *Hint:* Consider $f_n \in X$ equal to 0 at 0 and on [2/n,1], equal to n at $1/n$, linear otherwise.

Definition. X is normal if Hausdorff and for all $A, B \subset X$ closed, disjoint, there exist $U \supset A, V \supset B$ open and disjoint.

11. Any metrizable space is *normal*.

12. The Moore halfplane is regular, but not normal.

The *zero-one extension problem*: Given two disjoint closed sets $A, B \subset X$, there exists $f : X \to [0,1]$ continuous, so that $f \equiv 0$ on A, $f \equiv 1$ on B. *Urysohn's lemma* says this problem is solvable on any normal space.

13. Conversely, if the 0-1 extension problem is solvable for any two disjoint closed sets $A, B \subset X$, then X is normal.

14. The 0-1 extension problem is solvable in any metrizable space.

15. X is normal iff X is Hausdorff and for any $A \subset X$ closed, and any $U \supset A$ open, we may find $V \subset X$ open, so that

$$A \subset V \subset \overline{V} \subset U.$$

16. A regular, second countable space is normal.

17. (i) Subspaces of regular spaces are regular.
(ii) A product $X = \Pi_{\alpha \in \Lambda} X_\alpha$ is regular iff each X_α is.

18. Closed subspaces of normal spaces are normal. *Remark:* Products of normal spaces are not normal in general.

19. Let X be a normal space, $C \subset X$ closed, $f : C \to \mathbb{R}$ continuous. Use Tietze's extension theorem and a homeomorphism from $\mathbb{R}$ to $(-1, 1)$ to show f admits a continuous extension $F : X \to \mathbb{R}$.