Exercise 1

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Theorem. *If (M, d) is a complete metric space and $A \subseteq M$, then A is completely metrizable if and only if A is a G_δ subset of M.*

Complete the details of this outline of the proof:

(i) The graph G of ϕ is a closed subset of $M \times \mathbb{R}$ with the metric

$$\rho((x,t),(y,s)) = d(x,y) + |t - s|$$

for $x, y \in M$, $s, t \in \mathbb{R}$. (Hint: $G = \{(x,t) \mid tf(x) = 1\}$.)

(ii) The projection $M \times \mathbb{R} \to M$ restricted to G:

$$\begin{aligned} p : G &\to A \\ (x, \phi(x)) &\mapsto x \end{aligned}$$

is a homeomorphism from G to A. Note that the metric ρ is complete, so this shows that A is completely metrizable. An explicit complete on A is given by

$$d_1(x,y) = d(x,y) + \left| \frac{1}{d(x, A^c)} - \frac{1}{d(y, A^c)} \right|$$

where $A^c = M \setminus A$.

Proof. (i) Following the hint, note that

$$G = \{(x,t) : t = \phi(x) = 1/f(x)\} = \{(x,t) : tf(x) = 1\}.$$

Since we are in a metric space, a set is closed if and only if it is sequentially closed, so take $\{(x_n, t_n)\} \subseteq G$ be a sequence so that $(x_n, t_n) \to (x, t)$ with respect to ρ.

We show that $t = 1/f(x)$, so $(x,t) \in G$. Since $(x_n, t_n) \in G$, for each n, $t_n f(x_n) = 1$. Since f is continuous (metrics are continuous), $f(x_n) \to f(x)$ in $|\cdot|$. Furthermore, we have that $t_n \to t$ in d by the construction of ρ. Therefore $t_n f(x_n) \to tf(x)$, but $t_n f(x_n) = 1$ for all n, so we must have that $tf(x) = 1$. Hence $(x,t) \in G$, showing that G is closed.

Easier: $tf(x)$ is a continuous function, and so G is the inverse image of the set $\{1\}$, which is closed. Therefore G is closed.

(ii) We show that p is a homeomorphism. First we show that p is 1-1.

Let $(x, \phi(x)), (y, \phi(y)) \in G$ such that $p((x, \phi(x))) = p((y, \phi(y)))$. By definition of p, $x = y$, so we have that p is 1-1. Next we show that f is onto.

Let $y \in A$. Then $(y, \phi(y)) \in G$ and $p((y, \phi(y))) = y$, so p is onto. Now we show that p is continuous.

Since we are in a metric space, p is continuous if and only if it is sequentially continuous, so let $\{(x_n, \phi(x_n))\} \subseteq G$ be a sequence such that $(x_n, \phi(x_n)) \to (x, \phi(x))$. Since G is closed, $(x, \phi(x)) \in G$ also. Then we must show that $p((x_n, \phi(x_n))) \to p((x, \phi(x)))$, which is the same as showing that $x_n \to x$. This is of course true by the definition of ρ. Lastly we show that p is closed.

Let $\{x_n\} \subseteq A$ be a sequence such that $x_n \to x \in A$. We must show that $p^{-1}(x_n) \to p^{-1}(x)$. By definition of p,

$$p^{-1}(x_n) = (x_n, \phi(x_n)) \to (x, \phi(x)) = p^{-1}(x)$$

where the convergence comes from the continuity of ϕ and definition of ρ. Therefore p is a homeomorphism between G and A.

Noting that ρ is complete because d and $|\cdot|$ are, A is completely metrizable as it is a space homeomorphic to a completely metrizable space. The explicit metric d_1 defined above is just rewriting ρ in terms of only elements of A using the fact that $t = \phi(x)$ for $(x, t) \in G$, and

$$\phi(x) = \frac{1}{d(x, M \setminus A)}.$$

□