MATH 561 - PROPER METRIC SPACES

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Exercise 1. Let (X, d) be a proper metric space. Then X is complete, locally compact, and σ-compact.

Solution. Let (x_n) be a Cauchy sequence in X. Then there exists some $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m > N$. Let $n_0 > N$. Then if $\overline{B}(x_{n_0}, 1)$ is the closed ball centered at x_{n_0} with radius 1, it follows that $x_n \in \overline{B}(x_{n_0}, 1)$ for all $n > N$. Since X is proper, $\overline{B}(x_{n_0}, 1)$ is compact, and thus sequentially compact. Hence $(x_n)_{n>N}$ contains a convergent subsequence. Then since $(x_n)_{n>N}$ is Cauchy, it converges, and so too does (x_n). Therefore (X, d) is complete.

Let $x \in X$ and let U be an open neighborhood of x. Then for some $\epsilon > 0$, the closed ball $\overline{B}(x, \epsilon)$ is contained in U. Since (X, d) is proper, $\overline{B}(x, \epsilon)$ is compact. Thus $\overline{B}(x, \epsilon)$ is a compact neighborhood of x contained in U. Therefore X is locally compact.

Fix $x \in X$. We have that $X = \cup_{n \in \mathbb{N}} \overline{B}(x, n)$ where each $\overline{B}(x, n)$ is compact by the fact that (X, d) is proper. Hence X is σ-compact.

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