MATH 561 - HOMEWORK 4

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2. Given a countable collection of metric spaces (M_i, d_i) define the metric on $M = \prod_{i=1}^{\infty} M_i$ by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}.$$

As we know, $\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$ and $d_i(x_i,y_i)$ are equivalent metrics on M_i. Prove that (M,d) is complete if and only if each (M_i,d_i) is complete.

Solution. Suppose (M,d) is complete. Let $i \in \mathbb{N}$ and let (x^n) be a Cauchy sequence in (M_i,d_i). Choose points $x_j \in M_j$ for all $j \neq i$. Let $f : (M_i,d_i) \to (M,d)$ be the embedding which sends $x \in M_i$ to the point in M whose i-th coordinate is x and whose j-th coordinate is x_j for all $j \neq i$. Since $d(f(x),f(y)) = \frac{1}{2^i} \cdot \frac{d_i(x,y)}{1+d_i(x,y)} < d_i(x,y)$ for all $x, y \in M_i$, we have that $(f(x^n))$ is a Cauchy sequence in M. Hence it converges to a point x'. Then since $p_i(f(x^n)) = x^n$, where $p_i : (M,d) \to (M_i,d_i)$ is the projection, we have that (x^n) converges to $p_i(x')$. Therefore (M_i,d_i) is complete. Since $i \in \mathbb{N}$ was arbitrary, (M_i,d_i) is complete for all $i \in \mathbb{N}$.

Now in order to prove the other direction, we first show that the projection maps $p_i : (M,d) \to (M_i,d_i)$ are uniformly continuous. Let $i \in \mathbb{N}$ and let $\epsilon > 0$. Set $\delta = \frac{1}{2^i} \cdot \frac{\epsilon}{1+\epsilon}$. Then for any $x, y \in M$ such that $d(x,y) < \delta = \frac{1}{2^i} \cdot \frac{\epsilon}{1+\epsilon}$, we have that

$$\frac{1}{2^i} \cdot \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{d_j(x_j,y_j)}{1+d_j(x_j,y_j)} = d(x,y) < \frac{1}{2^i} \cdot \frac{\epsilon}{1+\epsilon}.$$

Then by mulitplying both sides of the inequality by $2^i(1+\epsilon)(1+d_i(x_i,y_i))$ and then subtracting $\epsilon \cdot d_i(x_i,y_i)$ we have that

$$\frac{1}{2^i} \cdot \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} < \frac{1}{2^i} \cdot \frac{\epsilon}{1+\epsilon} \implies d_i(x_i,y_i) < \epsilon.$$

Thus p_i is uniformly continuous for all $i \in \mathbb{N}$.

Suppose each (M_i,d_i) is complete. Let (x^n) be a Cauchy sequence in (M,d). Since p_i is uniformly continuous for all i, then (x_i^n) is Cauchy and hence converges to a point, x_i, for all $i \in \mathbb{N}$. Set $x = (x_i)$. We show that (x^n) converges to x. Let $\epsilon > 0$. Find $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \epsilon$. Let $\epsilon' = \epsilon - \frac{1}{2^m}$. For all $i \leq m$ find N_i such that $d_i(x_i^n, x_i) < \epsilon'$ for all $n \geq N_i$. Set $N = \max\{N_i \mid i \leq m\}$. Then for all $n \geq N$,

$$d(x,x^n) = \sum_{i=1}^{m} \frac{1}{2^i} \cdot \frac{d_i(x_i,x_i^n)}{1+d_i(x_i,x_i^n)} + \sum_{i=m+1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(x_i,x_i^n)}{1+d_i(x_i,x_i^n)} < \epsilon' + \frac{1}{2^m} = \epsilon.$$

Hence (x^n) converges to x and therefore (M,d) is complete.

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