Tariq Islam

Math 561

Proper Metric Space

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Exercise 2.

A metric space (X, d) is proper if and only if the distance function to a point $x \mapsto d(x, x_0)$ is a proper function (preimage of compact is compact.)

Def : (X, d) is proper (or HB) if closed bounded sets are compact.

Distance function $d : X \to \mathbb{R}$ defined by $d(x) := d(x, x_0)$.

Proof :

($\Rightarrow$) Suppose (X, d) is proper. *i.e.* closed bounded sets are compact in X.

To show: d is proper. *i.e.* preimage of compact sets are compact.

Let C be a compact set in $\mathbb{R}$. C is closed and bounded in $\mathbb{R}$.

Since d is continuous, $d^{-1}(C)$ is closed.

Also, let $x_1, x_2 \in d^{-1}(C)$, then $d(x_1, x_2) \leq \underbrace{d(x_1, x_0)}_{\in C} + \underbrace{d(x_2, x_0)}_{\in C} < \infty$

So, $d^{-1}(C)$ is bounded.

Therefore, $d^{-1}(C)$ is compact by hypothesis.

($\Leftarrow$) Suppose d is proper.

To show : (X, d) is proper.

Let $A \subset X$ be closed and bounded.

Claim : $d(A) \subset \mathbb{R}$ is closed [or, $\mathbb{R} \setminus d(A)$ is open.]

Let $y \in \mathbb{R} \setminus d(A)$.

Let U_y be an open neighborhood of y such that $\overline{U_y}$ is compact. (Since $\mathbb{R}$ is locally compact).

Suppose $U_y \cap d(A) \neq \emptyset$, if $U_y \cap d(A) = \emptyset$, then we are done.

Now $d^{-1}(\overline{U_y})$ is compact in X. (since d is proper)

Then, $A \cap d^{-1}(\overline{U_y})$ is also compact. (closed subset of a compact set.)

So, $d\Big(A \cap d^{-1}(\overline{U_y})\Big) = d(A) \cap \overline{U_y} \subset \mathbb{R}$ is compact (d continuous) and in particular closed (compact subset of $\mathbb{R}$ Hausdorff)

Then $V_y := U_y \setminus \Big(d(A) \cap \overline{U_y}\Big) = U_y \setminus d(A)$ is an open neighborhood of y such that $V_y \cap d(A) = \emptyset$.

Therefore, $\mathbb{R} \setminus d(A)$ is open.

Then $d(A)$ is a closed and bounded set in $\mathbb{R}$, hence compact.

So, $d^{-1}\big(d(A)\big)$ is compact, since d is proper.

Now, $A \subset d^{-1}\big(d(A)\big)$ is a closed subset of a compact set, hence compact.

Therefore, (X, d) is proper.