Proper Metric Spaces: Exercise 3

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Theorem. *If (X, d) is a locally compact and σ-compact, then there exists an equivalent metric which is Heine-Borel.*

Idea of Proof.

Let $\{K_n\}$ be a compact exhaustion of X: $K_n \subseteq \text{int}(K_{n+1})$. Let $f_n : X \to [0, 1]$ continuous satisfy: $f_n \equiv 0$ on K_n, $f_n \equiv 1$ on K_{n+1}^c. Consider the metric on X:

$$d'(x, y) = d(x, y) + \sum_{n \geq 1} |f_n(x) - f_n(y)|.$$

Exercise 3.

Show that d and d' are equivalent on X. (It suffices to show that $x_n \to x$ in d if and only if $x_n \to x$ in d'.)

Proof. One direction is easy: If $x_n \to x$ in d', then $d'(x_n, x) \to 0$, so we necessarily have that $d(x_n, x) \to 0$, so $x_n \to x$ in d.

For the other direction, if $x_n \to x$ in d, we just have to show that

$$\sum_{i \geq 1} |f_i(x_n) - f_i(x)| \to 0$$

as $n \to \infty$.

If $x \in K_i$, then $\exists U \subseteq K_i$ such that $x \in U$, so eventually $x_n \in U \subseteq K_i$.

If instead $x \in K_{i+1}^c$, then K_{i+1}^c is open, so eventually $x_n \in K_{i+1}^c$.

As $\{K_i\}$ is a compact exhaustion of X, one of these two will occur for some i, so we have that

$$\sum_{i \geq 1} |f_i(x_n) - f_i(x)| = 0$$

for sufficiently large n.

Therefore d and d' are equivalent metrics on X. $\square$