TOPOLOGY

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Proof that continuous functions from a subset of a metric space to another metric space allow a continuous extension to a certain subset of the closure of that subset in the domain:

Exercise 4) Let X and Y be metric spaces with metrics d_X and d_Y, $A \subset X$, and $f : A \to Y$ a continuous function. Define

$$G = \left\{ a \in \bar{A} : \lim_{x \to a} f(x) \text{ exists} \right\},$$

and $g : G \to Y$ by $g(a) = \lim_{x \to a} f(x)$. Clearly, since f is continuous, $A \subset G$ and $g|_A = f$, so we need only show g is continuous to show it is continuous extension of f to G. Let $\varepsilon > 0$, and let $a \in G$ and choose $\delta > 0$ so $d_Y(g(a), f(b)) < \varepsilon$ if b is an element of A so $d_X(a, b) < \delta$. Certainly, such a δ exists because $g(a)$ is the limit of f at a. Now, if x is in A and within δ distance of a in X, then by definition $d_Y(g(a), g(x)) = d_Y(g(a), f(x)) < \varepsilon$. If instead $x \in G \setminus A$ is so $d_X(a, x) < \delta$ then we must have $d_Y(g(a), g(x)) \leq \varepsilon$.

To see this, suppose instead that $d_Y(g(a), g(x)) > \varepsilon$. Then, there exists some δ_0 so that for any $b \in A$, $d_X(x, b) < \delta_0$ implies both that $d_Y(g(x), f(b)) < d_Y(g(a), g(x)) - \varepsilon$ and $d_X(a, b) < \delta$. But then, b is so

$$d_Y(g(a), f(b)) \geq d_Y(g(a), g(x)) - d_Y(g(x), f(b)) > \varepsilon$$

but since $d_X(a, b) < \delta$ then we know $d_Y(g(a), f(b)) < \varepsilon$, a contradiction. Thus, we have shown that for any $x \in G$, $d_X(a, x) < \delta$ implies $d_Y(g(a), g(x)) \leq \varepsilon$. Since $a \in G$ was arbitrary, g is continuous on G.