Topology Arzela-Ascoli: Exercise 2

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Exercise 2

Let (X, d) be a metric space. If a sequence $(fn)_{n\geq 1}$ of functions in $C_E(X)$ converges to $f \in C_E(X)$ uniformly on X, then the family $\mathcal{F} = \{f1, f2, \ldots, fn, ...f\}$ is equicontinuous at each $x_0 \in X$.

Proof. Let (X, d) be a metric space and let $(fn)_{n\geq 1}$ be a sequence converging to $f \in C_E(X)$ uniformly on X. Thus, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(f_n(x), f(x)) < \frac{\epsilon}{3}$$

Furthermore, since f is continuous, we have that for any $\epsilon > 0$, there exists $\delta_f > 0$ such that

$$d(x, x_0) < \delta_f \implies d(f(x), f(x_0)) < \frac{\epsilon}{3}$$

Also, each f_n is continuous, so for each $n \leq N$ and for all $\epsilon > 0$, there exists $\delta_n > 0$ such that

$$d(x, x_0) < \delta_n \implies d(f_n(x), f_n(x_0)) < \frac{\epsilon}{3}$$

Now, let $\delta = min\{\delta_1, \delta_2, \ldots, \delta_N, \delta_f\}$. If $n \leq N$, we are done because clearly $\delta \leq \delta_n$. If $n > N$, then assume $d(x, x_0) < \delta$. We can then use the triangle inequality as follows.

$$d(f_n(x), f_n(x_0)) \leq d(f_n(x), f(x)) + d(f(x) + f(x_0)) + d(f(x_0), f_n(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Since this holds for all $f_i \in \mathcal{F}$, we conclude $\mathcal{F}$ is equicontinuous at each $x_0 \in X$ $\square$