Topology HW 1

Elisha Brooks

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10. In a metric (or metrizable) space X, $E \subset X$ is closed iff E is sequentially closed.

<u>Proof</u>:
$(\implies)$
First, assume X is a metric space, $E \subset X$, and E is closed. Since E is closed, we know that $E = \overline{E}$. So let $x \in E$ and d be a metric for the topology on X. Then $\forall n \in \mathbb{N}$, we can take the neighborhood at x $B_d(x, \frac{1}{n})$. Let x_n be a point of this neighborhood's intersection with E, which we know exists by the definition of closure.
Any open set U containing x contains an ϵ-ball $B_d(x, \epsilon)$ centered at x, so we choose N above such that $\frac{1}{N} < \epsilon$. Then U contains x_i $\forall i \geq N$. This produces the sequence (x_i) in E converging to x, so E is sequentially closed.
$(\impliedby)$
We assume now that X is is a metric space, $E \subset X$, and E is sequentially closed (i.e. every convergent sequence in E converges to a limit in E).
Assume by contradiction that $E \neq \overline{E}$, i.e. there's a point $x \notin E$ that is a limit of E. We can use the metric on X to construct neighborhoods at x: $\forall n \in \mathbb{N}$, consider the neighborhoods $B_d(x, \frac{1}{n})$. Since $x \in E'$, every one of these neighborhoods must intersect E at a point other than x; in other words, $B(x, \frac{1}{n}) \cap (E - x)$ is nonempty. Choosing a point in each of these intersections, we have a sequence (x_n) converging to x. Because E is sequentially closed, we have a contradiction. Thus $x \in E$ and E must be closed.