Problem Set 1

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$(X, d), E \subset X, \neq X, d(x, E) := \inf\{d(x, y); y \in E\}$.

(i) $f(x) = d(x, E)$ is continuous on X (and Lipschitz).

(ii) $d(x, E) = d(x, \overline{E})$.

Proof:

Let (X, d) be a metric space. Let $E \subsetneq X$. Define $d(x, E) := \inf\{d(x, y); y \in E\}$.

(i) Let $f : X \to \mathbb{R}$ be defined by $f(x) := d(x, E)$. To show that $f(x)$ is continuous on X.

To show $f(x)$ **Lipschitz**, we need to show $\forall x, z \in X$ $\exists M > 0$ such that $|f(x) - f(z)| \leq M|x - z|$ $i.e$ $|d(x, E) - d(z, E)| \leq M d(x, z)$

For any $y \in E$, $d(x, E) \leq d(x, y)$, and
by definition of inf, $\exists y_n \in E$ such that $d(z, E) \geq d(z, y_n) - \frac{1}{n}$, $n \in \mathbb{N}$

Then $d(x, E) - d(z, E) \leq d(x, y_n) - d(z, y_n) + \frac{1}{n} \leq d(x, z) + \frac{1}{n}$
Now, if we let $n \to \infty$, we have
$d(x, E) - d(z, E) \leq d(x, z)$
Similarly by switching x and z we get, $d(z, E) - d(x, E) \leq d(x, z)$

Then $|d(x, E) - d(z, E)| \leq d(x, z)$ implies $f(x)$ Lipschitz.
Therefore, f is continuous on X.

(ii) Define $d(x, \overline{E}) := \inf\{d(x, y'); y' \in \overline{E}\}$.

$d(x, \overline{E}) = \inf\{d(x, y'); y' \in \overline{E}\} \leq \inf\{d(x, y); y \in E\} = d(x, E)$, since $E \subseteq \overline{E}$.

On the other hand, for any $y' \in \overline{E}$, $\forall \varepsilon > 0$, $\exists z \in E$ such that $d(z, y') < \varepsilon$.

Then, $d(x, E) = \inf\{d(x, y); y \in E\} \leq d(x, z) \leq d(x, y') + d(y', z) < d(x, y') + \varepsilon$
Since ε is arbitrary, we have $d(x, E) \leq d(x, y')$, $\forall y' \in \overline{E}$

Now, by taking the inf over $y' \in \overline{E}$, we have $d(x, E) \leq d(x, \overline{E})$
Therefore, $d(x, E) = d(x, \overline{E})$