1 Quasi Isometric metrics

Definition 1.1. d_1 and d_2 are quasi-isometric if $\exists A > 1$ such that

$$\frac{1}{A}d_2(x,y) \leq d_1(x,y) \leq Ad_2(x,y)$$

- Two quasi-isometric metrics define the same topology(i.e. are equivalent)

  Proof. Let d_1 and d_2 be two quasi-isometric metrics on X with τ_1 and τ_2 being two topologies generated by the metrics respectively. We want to show that $\tau_1 = \tau_2$. We know that the set of all open balls in (X, d_1) form a basis for τ_1 and likewise , the set of all open balls in (X, d_2) form a basis for τ_2. It would suffice to show that any d_1 open ball contains a d_2 open ball with the same center and vice-versa.
  Let $x \in X$ be given, we want to show that $\forall \epsilon > 0 \ \exists \delta_\epsilon > 0$ such that

  $$B_{d_2}(x, \delta_\epsilon) \subset B_{d_1}(x, \epsilon)$$

  i.e. For $y \in X$, $d_2(x,y) < \delta_\epsilon \to d_1(x,y) < \epsilon$

  Choose $\delta_\epsilon = \epsilon/A$, By definition,

  $$d_1(x,y) \leq Ad_2(x,y) < A \cdot .\frac{\epsilon}{A} = \epsilon$$

  The converse is identical(By using the left half of the definition, and swapping d_1 and d_2 in the proof above) □

- $d_1(x,y) = |x-y|$ and $d_2(x,y) = |x^3 - y^3|$ define equivalent metrics on $\mathbb{R}$ which are not quasi-isometric

  - Not quasi-isometric.

    If both metrics are quasi-isometric, then they satisfy the inequality:

    $$\frac{1}{A}|x-y| \leq |x-y||x^2+xy+y^2| \leq A|x-y|$$

    If $x = y$, then any $A > 1$ satisfies the equation. On the other hand, if $x \neq y$,we need to show that there exists $A > 1$ such that

    $$\frac{1}{A} \leq |x^2+xy+y^2| \leq A$$

    For $x_n, y_n \to \infty$, $|{x_n}^2 + x_n y_n + {y_n}^2|$ is not bounded. Therefore, no such A exists for all $x, y \in X$.

  - Equivalence

    Let $x \in \mathbb{R}$ be given, we want to show that $\forall \epsilon > 0 \ \exists \delta > 0$ such that

    $$B_{d_1}(x, \delta) \subset B_{d_2}(x, \epsilon)$$

    i.e. For $y \in \mathbb{R}$, $|x-y| < \delta \to |x^3 - y^3| < \epsilon$

Choose $\delta = \min\left(1, \frac{\epsilon}{3|x|^2+3|x|+1}\right)$

For $0 < |x-y| < \delta$, we can see that $|y| < 1 + |x|$. It follows then that:

$$\begin{aligned} |x^3 - y^3| &= |x-y||x^2+xy+y^2| \\ &< \delta|x^2+xy+y^2| \leq \delta(|x|^2 + |x||y| + |y|^2) < \delta(3|x|^2 + 3|x| + 1) < \epsilon \end{aligned}$$