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Math 561

Problem Set 2

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Problem15

X is normal iff X is Hausdorff and for any $A \subset X$ closed, and any $U \supset A$ open, we may find $V \subset X$ open, so that $A \subset V \subset \overline{V} \subset U$.

Def : X is normal if Hausdorff and $\forall A, B \subset X$ closed, disjoint, $\exists U, V$ open neighborhoods of A, B respectively, s.t. $U \cap V = \emptyset$.

Proof :
($\Rightarrow$) Suppose X is normal. Then by definition X is Hausdorff.

Let $A \subset X$ be closed and $U \subset X$ be an open cover of A.

Then U^c is closed in X.

Since $A, U^c \subset X$ are closed and disjoint, $\exists V, W \subset X$ open s.t. $A \subset V$, $U^c \subset W$, and $V \cap W = \emptyset$.

Claim: $\overline{V} \subset U$

Clearly $V \subset U$, since V and W are disjoint and W contains U^c.
ETS: $V' \subset U$.
Let $x \in V'$, then every neighborhood of x intersects V in some point other than x itself.

Which means $x \notin U^c$. If x was in U^c, then since V and U^c are disjoint, for any $y \in V$ $x \neq y$. So, using the fact that X is Hausdorff, we can find disjoint open neighborhood around y and x, which contradicts the fact that $x \in V'$.
So, $V' \in U$.

($\Leftarrow$) Let $A, B \subset X$ closed and disjoint.

Then A^c is open and $B \subset A^c$.

By hypothesis, $\exists V$ open such that $B \subset V \subset \overline{V} \subset A^c$.

Then $U = \overline{V}^c$ is open in X and $A \subset U$ and $U \cap V = \emptyset$.

Therefore, X is normal.