Problem 2.19

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Let X be a normal space, $C \subseteq X$ be a closed set, and $f : C \to \mathbb{R}$ a continuous function. Use Tietze's extension theorem and a homeomorphism from $\mathbb{R}$ to $(-1, 1)$ to show that f admits a continuous extension $F : X \to \mathbb{R}$.

Proof. To begin, let $g : \mathbb{R} \to (-1, 1)$ be a homeomorphism. E.g.

$$g(x) = \frac{2}{\pi} \arctan(x).$$

Then $(g \circ f) : C \to (-1, 1)$ is a continuous function, so by Tietze's extension theorem, there is a continuous extension $G : X \to [-1, 1]$. To show what we want, we have to make sure that G does not take the values -1 and 1.

Consider the set $A = G^{-1}(\{-1, 1\})$. Since G is continuous and $\{-1, 1\}$ is closed in $[-1, 1]$, A is closed, as it is the inverse image of a closed set under a continuous function. Further, C is closed by assumption, and A and C are disjoint. To see this, $G(C) \subseteq (-1, 1)$ since it is an extension of $g \circ f$, but $G(D) = \{-1, 1\}$.

Since X is normal, we can use Urysohn's lemma to find a function $h : X \to [0, 1]$ with $h(A) = \{0\}$ and $h(C) = \{1\}$. Now consider $(G \cdot h) : X \to [-1, 1]$. For $x \in C$,

$$(G \cdot h)(x) = G(x) \cdot h(x) = G(x) = (g \circ f)(x),$$

so it is still an extension of $g \circ f$. For $x \in A$,

$$(G \cdot h)(x) = G(x) \cdot h(x) = \pm 1 \cdot 0 = 0.$$

Hence $(G \cdot h)(X) \subseteq (-1, 1)$. Since G and h are both continuous, $G \cdot h$ is continuous. Now define

$$F = g^{-1} \circ (G \cdot h).$$

Then $F : X \to \mathbb{R}$ is the desired continuous extension of f. □