Topology HW 2

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4. $(H, halfdisk)$, the upper half plane with the half-disk topology, is Hausdorff but not regular.

Proof:
We are working in $H = P \cup L$: the upper half plane $P = \{(x, y)|x, y \in \mathbb{R}, y > 0\}$, unioned with the x-axis L.
To see that this space is Hausdorff is trivial. For any two points in the plane, we can see that there exist disjoint open balls centered on each. For a point on the line L and a point in the plane, we can do the same thing as above. For any two points on the line L, we can take any open half-disk neighborhood centered at each and shrink their diameters until they are disjoint.

By definition, regular means that in our topological space, given any point x and a disjoint closed set B, we can find disjoint open sets containing x and B respectively.
Let $x \in L$ and consider a subbasis element centered at x, D_x. If we take $C = H - D_x$, we can see that C contains all of the diameter points of D_x except for x itself.
Now take some open neighborhood of x, V. $\overline{V}$ must intersect C (i.e. at those diameter points). In other words, every neighborhood of the closed set C intersects every neighborhood of x. This shows that the space is not regular.