# Topology Problem 10

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In the linear space $X = C[0,1]$ of continuous, real-valued functions on $[0,1]$, consider the supremum and $L^1$ norms:

$$\|f\|_{\sup} := \sup\{|f(t)| : t \in [0,1]\}$$

and

$$\|f\|_{L^1} := \int_0^1 |f(t)| \, \mathrm{d}t.$$

(i) Any $L^1$ ball contains a supremum ball:

$$B_{L^1}(f_0, R) \supseteq B_{\sup}(f_0, R), \qquad \forall f_0 \in X, \, \forall R > 0.$$

(ii) The $L^1$ ball $B_{L^1}(0,1)$ is not contained in any ball $B_{\sup}(0,R)$. Thus these two norms define different topologies on $X$.

(i) Let $g \in B_{\sup}(f_0, R)$. Then $\|g - f_0\|_{\sup} < R$, meaning

$$\sup_{t\in[0,1]} |g(t) - f_0(t)| < R.$$

We must show that $g \in B_{L^1}(f_0, R)$, i.e.

$$\int_0^1 |g(t) - f_0(t)| \, \mathrm{d}t < R.$$

We have

$$\begin{aligned}
\int_0^1 |g(t) - f_0(t)| \, \mathrm{d}t &\leq \int_0^1 \sup_{t\in[0,1]} |g(t) - f_0(t)| \, \mathrm{d}t \\
&= \sup_{t\in[0,1]} |g(t) - f_0(t)| \int_0^1 \mathrm{d}t \\
&= \sup_{t\in[0,1]} |g(t) - f_0(t)| < R.
\end{aligned}$$

Hence $B_{L^1}(f_0, R) \supseteq B_{\sup}(f_0, R)$. $\square$

(ii) It suffices to show that we can find continuous functions on $[0,1]$ which integrate to less than 1, but are unbounded as a collection. Take

$$f_n(t) := \begin{cases} \frac{n^2 t}{2}, & t \in \left[0, \frac{1}{n}\right], \\ n - \frac{n^2 t}{2}, & t \in \left[\frac{1}{n}, \frac{2}{n}\right], \\ 0, & t \in \left[\frac{2}{n}, 1\right]. \end{cases}$$

Figure 1 shows the plot of this:

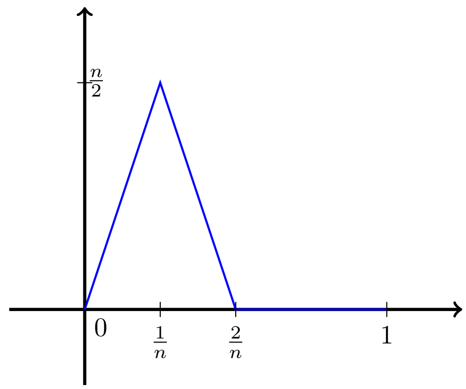


Figure 1: The plot of $f_n$.

We have

$$\begin{aligned}\int_0^1 f_n(t)\,\mathrm{d}t &= \int_0^{1/n} \frac{n^2 t}{2}\,\mathrm{d}t + \int_{1/n}^{2/n} \left(n - \frac{n^2 t}{2}\right)\,\mathrm{d}t + \int_{2/n}^{1} 0\,\mathrm{d}t \\ &= \left[\frac{n^2t^2}{4}\right]_0^{1/n} + \left[nt - \frac{n^2t^2}{4}\right]_{1/n}^{2/n} \\ &= \frac{1}{4} - 1 + 2 + \frac{1}{4} - 1 = \frac{1}{2} < 1.\end{aligned}$$

So $\|f_n\|_{L^1} < 1$, hence $f_n \in B_{L^1}(0,1)$ for all $n$. But we also have

$$\sup_{t\in[0,1]} |f_n(t)| = \frac{n}{2},$$

which means that no matter what $R > 0$ we choose, there would be an $n$ with $f_n \notin B_{\sup}(0,R)$. Hence $B_{L^1}(0,1)$ does not contain any supremum ball, so these norms define different topologies on $C[0,1]$. $\square$