PS2.6

Claim
If X, Y are metric spaces, Y is complete, $A \subset X$ and $f : A \mapsto Y$ is uniformly continuous on A, then there is a unique extension of f to a continuous map $F : \overline{A} \mapsto Y$.
Proof
Let $(a_n) \in A$ Cauchy with $(a_n) \mapsto a \in \overline{A}$. Define $F : \overline{A} \mapsto Y$ as $F(a) = \lim_{n\to\infty} f(a_n)$. Then clearly $F(a)$ exists in Y since it is complete and all Cauchy sequences–(a_n) is thus $f(a_n)$ is by virtue of f being uniformly continuous–converge inside Y. Moreover, its well-defined since if $(a_n), (\alpha_n)$ both converge to a, we can concatenate them: $(x_n) = (a_1, \alpha_1, a_2, \alpha_2, ...)$, and assuming they are individually Cauchy, still take a sufficiently large $n \in N$ such that, etc. Then $(x_n) \mapsto a$, and as $(a_n), (\alpha_n)$ are subsequences of (x_n), $(f(a_n)), (f(\alpha_n))$ are subsequences of $(f(x_n))$ so their limits are the same.

Now we claim F is uniformly continuous:

Let $a, b \in \overline{A}$ with $d_X(a, b) < \delta/3$. For $(a_n), (b_n) \in A$ converging to a, b respectively, we can take a sufficently large $n_1 \in N$ such that $d_x(a_n, a) < \delta/3$ as well as $d_x(b_n, b) < \delta/3$ for $n > n_1$.

Then $d_X(a_n, b_n) \leq d_x(a_n, a) + d_X(a, b) + d(b, b_n) < \delta$ for $n > n_1$. Then we have $\mathrm{d}_Y(f(a_n), f(b_n)) < \epsilon/3$.

We also have $f(a_n) \longrightarrow F(a)$ and $f(b_n) \longrightarrow F(b)$, so there is $n_2 \in N$ such that for $n > n_2$ we have $\mathrm{d}_Y(f(a_n), F(a)) < \epsilon/3$ and $d_y(f(b_n), F(b)) < \epsilon/3$.

Taking $N = max\{n_1, n_2\}$, we have $d_Y(F(a), F(b)) \leq d_y(F(a), f(a_N)) + d_Y(f(a_N), f(b_N)) + d_Y(f(b_N), F(b)) < \epsilon$ as required.

Moreover, F is unique. Suppose $G : \overline{A} \mapsto Y$ is a continuous extension of f. Let $a \in \overline{A}$ with $(a_n) \longrightarrow a$. Because G is continuous, we have $G(a_n) \longrightarrow G(a)$. Then we have $G(a_n) = f(a_n)$ as well as $f(a_n) \longrightarrow F(a)$. Then we have obtained $G(a) = F(a)$ So G and F are the same map.