Problem 7: The continuous surjective image of a separable space is separable. That is, if $f : X \to Y$ is continuous and onto, and X is separable, then so is Y.

Proof. Since X is separable, there exists some countable subset $M \subset X$ such that $\overline{M} = X$. Observe that $f(M)$ is also countable. We claim that $\overline{f(M)} = Y$. It is clear that $\overline{f(M)} \subset Y$ so it is sufficient to show that $Y \subset \overline{f(M)}$.

Let $y \in Y$, and let V be an open set that contains y. Continuity of f implies that $f^{-1}(V)$ is open in X. Moreover, since f is surjective, there exists some $x \in f^{-1}(V)$ such that $f(x) = y$.

Now, recall Theorem 17.5 (a) from Munkres:

Theorem 17.5 (a). *Let A be a subset of the topological space X. Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.*

Since $x \in X = \overline{M}$, and $f^{-1}(V)$ is an open set that contains x, it follows from 17.5(a) that there exists some $m \in M$ such that $m \in f^{-1}(V)$. Thus, $f(m) \in V$. Observe that $f(m) \in f(M)$. Since our choice of V was arbitrary, we deduce that every open set of Y containing y must intersect $f(M)$. By 17.5(a), $y \in \overline{f(M)}$. Thus, $Y \subset \overline{f(M)}$.

□