Topology HW2

Bryan Walker

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8) Let C_L denote the vector space of Lipschitz continuous functions from $[0,1]$ to $\mathbb{R}$ and define, for any $f \in C_L$

$$[f] = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|},$$

which is guaranteed to be a finite number since f is Lipschitz. Then, we define $||f|| = |f(0)| + [f]$ and show that it is a norm on C_L. Both $|f(0)|$ and $[f]$ are positive, so $||f|| \geq 0$ and clearly $||0|| = 0$, so we show that $||f|| = 0$ implies $f = 0$.

Let $f \in C_L$ be so $||f|| = |f(0)| + [f] = 0$. Then, $f(0) = 0$ and, for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq [f]|x - y| = 0,$$

so $f(x) = f(y)$ and f is a constant function. But, since $f(0) = 0$, then f is the zero function.

The property $||cf|| = |c|||f||$ follows from the equation

$$\frac{|cf(x) - cf(y)|}{|x - y|} = \frac{|c||f(x) - f(y)|}{|x - y|},$$

and the triangle inequality $||f + g|| \leq ||f|| + ||g||$ follows from

$$\frac{|f(x) + g(x) - f(y) - g(y)|}{|x - y|} \leq \frac{|f(x) - f(y)|}{|x - y|} + \frac{|g(x) - g(y)|}{|x - y|},$$

from the triangle inequality for real numbers, so $|| \cdot ||$ is a norm on C_L. If we define, for any $t \in (0,1) \subset \mathbb{R}$, the function $f_t : [0,1] \to \mathbb{R}$ by

$$f_t(x) = \begin{cases} 0 & \text{if } x \leq t, \\ x - t & \text{if } x > t, \end{cases}$$

then each $f_t \in C_L$ with $||f_t|| = 1$. If $t_1 < t_2$, then

$$(f_{t_1} - f_{t_2})(x) = \begin{cases} 0 & \text{if } x \leq t_1, \\ x - t_1 & \text{if } t_1 < x \leq t_2, \\ t_2 - t_1 & \text{if } x > t_2, \end{cases}$$

so

$$||f_{t_1} - f_{t_2}|| = |f_{t_1}(0) - f_{t_2}(0)| + [f_{t_1} - f_{t_2}] = 1,$$

so the collection $\{f_t\}_{t \in (0,1)}$ is an uncountable collection of sets in C_L each a distance 1 apart, precluding any countable dense subset in C_L (take balls of radius 1/2 about each such f_t and note that they are all mutually disjoint so no countable set could intersect them all nontrivially).