MATH 561 - HOMEWORK 2

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9. Show that

(i) A discrete uncountable space cannot be second-countable

(ii) The Moore plane is seperable and first-countable, but not second-countable (hence not metrizable.)

Solution. (i) Suppose the singleton set $\{x\}$ is open in X for some $x \in X$ and $\mathscr{B}$ is a basis for the topology on X. Then $\{x\}$ can be written as the union of elements of $\mathscr{B}$, but this implies that $\{x\} \in \mathscr{B}$. Thus if a space X is discrete, and $\mathscr{B}$ is a basis for the topology on X, then $\{x\} \in \mathscr{B}$ for all $x \in X$. Hence if X is discrete and uncountable, then any basis $\mathscr{B}$ contains uncountably many elements as well, thus X is not second-countable.

(ii) Let $\mathscr{U}$ be the set of open balls contained in the open half plane $\{(x, y) \mid x, y \in \mathbb{R}\}$. Let $A = \{(p, q) \mid p, q \in \mathbb{Q}, q > 0\}$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, clearly, A is dense in the open half plane with the subspace topology inherited from $\mathbb{R}^2$ (which is equivalent to the topology generated by $\mathscr{U}$.) By definition, every set V which is open in the Moore plane contains some $U \in \mathscr{U}$. Thus A is dense in the Moore plane. Moreover, A is countable, hence the Moore plane is seperable.

Let (x, y) be a point in the Moore plane. If $y > 0$, then the set of open balls contained in the Moore plane with center (x, y) and radius $\frac{1}{n}$ for $n \in \mathbb{N}$ forms a countable local basis at (x, y). Similarly if $y = 0$, then the open sets $U_n \cup \{(x, y)\}$ where U_n is an open ball tangent to (x, y) with radius $\frac{1}{n}$ for $n \in \mathbb{N}$ form a countable local basis at (x, y). Thus every point in the Moore plane has a countable local basis, and so the Moore plane is first-countable.

However, the Moore plane is not second-countable. This is because the line $L = \{(x, 0) \mid x \in \mathbb{R}\}$ with the subspace topology from the Moore plane is discrete, thus by (i), any basis for$L$ is uncountable. Hence any basis $\mathscr{B}$ for the Moore plane is uncountable, since $\{B \cap L \mid B \in \mathscr{B}\}$ is a basis for L and is necessarily uncountable. □

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