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Math 561

Problem Set 1

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Problem 2

Let $Y \subset X$ have the induced topology. $C \subset Y$ is closed in Y iff $C = A \cap Y$, for some $A \subset X$ closed in X.

Proof:

Let X be a topological space and Y be a subspace of X.

($\Rightarrow$)
Suppose $C \subset Y$ is closed in Y.
Then $Y \setminus C$ is open in Y
So, $\exists\ U$ open in X s.t. $Y \setminus C = U \cap Y$.

Now, $X \setminus U$ is closed in X.
Since, $Y \setminus C = U \cap Y$, $C = (X \setminus U) \cap Y$

Let $X \setminus U = A$
Then, $C = A \cap Y$, for some $A \subset X$ closed in X.

($\Leftarrow$)
Suppose $C = A \cap Y$ for some $A \subset X$ closed in X
Then $X \setminus A$ is open in X.
So, $(X \setminus A) \cap Y$ is open in Y.

But $(X \setminus A) \cap Y = Y \setminus C$
Then $Y \setminus C$ is open in Y.
Therefore, C is closed in Y. $\square$

Problem 2.5

Let $E \subset Y \subset X$, where X is a topological space and Y has the induced topology. Then $\overline{E}^Y$ (the closure of E in the induced topology on Y) equals $\overline{E} \cap Y$, the intersection of the closure of E in X with the subset Y.

Proof:

Let X be a topological space and Y be a subspace of X.
Let $E \subset Y$.

To Show: $\overline{E}^Y = \overline{E} \cap Y$.

Since $\overline{E}^Y$ is closed in Y,
$\overline{E}^Y = C \cap Y$ for some closed set C of X. (by problem 2) ($\star$)
Then C contains E.
Since C is a closed set contains E, C contains $\overline{E}$ (by the definition of $\overline{E}$).
So, $\overline{E} \subset C$
Then, $\overline{E} \cap Y \subset C \cap Y$
Since $\overline{E}^Y = C \cap Y$ ($\star$), $\overline{E} \cap Y \subset \overline{E}^Y$.

To show the other inclusion, $\overline{E}$ is closed in X.
This implies, $\overline{E} \cap Y$ is closed in Y. (by problem 2)

Now $\overline{E}$ contains E implies $\overline{E} \cap Y$ contains E.
Since, $\overline{E} \cap Y$ is a closed set in Y that contains E, $\overline{E} \cap Y$ contains $\overline{E}^Y$. (by definition of $\overline{E}^Y$)
So, $\overline{E}^Y \subset \overline{E} \cap Y$. $\square$