Topology Problem Set 3

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Problem 10. Let $A_n = \{x \in X : \forall f \in \mathcal{F}, |f(x)| \leq n\}$. Clearly, $A_1 \subset A_2 \subset A_3 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = X$ since $\mathcal{F}$ is bounded pointwise. But, since X is clearly dense in itself, this implies that not all of the A_n are nowhere dense (for otherwise, by the Baire property of complete metric spaces, their union could not be a dense set). In particular, there is an A_C whose closure $\bar{A}_C$ has nonempty interior U. But,

$$A_C = \bigcap_{f \in \mathcal{F}} f^{-1}([-C, C]),$$

so A_C is closed, so $U \subset A_C = \bar{A}_C$, so U is equibounded by C: that is, for any $x \in U$ and $f \in \mathcal{F}$,

$$|f(x)| \leq C.$$