Problem Set 3

Problem13

Let X be normal $A \subset X$ non-empty. There exists $f \in C(X;[0,1])$ with $\{x \in X; f(x) = 0\} = A$ iff A is a closed G_δ subset of X.

Def : A is a "G_δ set" in X if A is the intersection of a countable collection of open sets of X.

Proof :

($\Rightarrow$) Let $f \in C(X;[0,1])$ and $A = \{x \in X; f(x) = 0\}$

$A = f^{-1}(\{0\}) = f^{-1}\Big(\bigcap_{n\geq 1}[0, \frac{1}{n})\Big) = \bigcap_{n\geq 1}\Big(f^{-1}[0, \frac{1}{n})\Big)$.

Since $[0, \frac{1}{n})$ is open in $[0,1]$ and f is continuous, $f^{-1}[0, \frac{1}{n})$ is open in X.

So, A is a G_δ subset of X.

Also, since $\{0\}$ is closed in $[0,1]$, A is closed in X.

($\Leftarrow$) Let A is a closed G_δ subset of X.

Suppose, $A = \bigcap_{n\geq 1} U_n$, U_n open in X.

We may assume that $U_n \supset U_{n+1}$ (If not we can take intersection of U_n and U_{n+1} and then relabel the intersection as U_{n+1}. This intersection is open and still contains A.).

Since U_n is open containing A, U_n^c is closed and $A \cap U_n^c = \emptyset$.

Then using Urysohn Lemma, let $f_n \in C(X;[0;1])$ such that $f_n \equiv 0$ in A, and $f_n \equiv 1$ in U_n^c.

Now, Let $f : X \to [0,1]$, defined by $f(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x)$.

Note that $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}|f_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, then by Weierstrass M-test, $f(x)$ converges absolutely and uniformly.

$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x) \geq 0$, since $f_n \in C(X;[0;1])$.

So, $f(x)$ is continuous with values in $[0,1]$.

Also, if $x \notin A$, then $x \in \bigcup_{n\geq 1} U_n^c \implies f_n(x) = 1 \implies f(x) > 0$.

Therefore, $A = \{x \in X; f(x) = 0\}$.