4 Strong Urysohn's Lemma

Let X be normal, $A, B \subset X$ closed disjoint. Show there exists $f \in C(X; [0,1])$ so that $f^{-1}(0) = A, f^{-1}(1) = B$ iff A and B are G_δ sets.

Proof. A G_δ set is a countable intersection of open sets.
$\rightarrow$
If such f exists, then $A = f^{-1}\left(\bigcap_{i=1}^{\infty} B_{\frac{1}{i}}(0)\right)$ and $B = f^{-1}\left(\bigcap_{i=1}^{\infty} B_{\frac{1}{i}}(1)\right)$ are closed G_δ sets.

Conversely, let A, B be closed G_δ sets. Because A, B are disjoint, according the previous problem, we can choose $f_A, f_B \in C(X, [0,1])$ such that

$$f_A^{-1}(0) = A, f_A \equiv 1 \text{ in B}$$

$$f_B^{-1}(0) = A, f_B \equiv 1 \text{ in A}$$

The function $f(x) = \dfrac{f_A(x)}{f_A(x) + f_B(x)}$ satisfies our conditions.
It is trivial to see that $f(x) = 0\ \forall x \in A$ and $f(x) = 1\ \forall x \in B$. We would show that for $x \notin A \cup B$, $f(x) \notin \{0, 1\}$.
From 13, since A is a G_δ set , $f_A(x) > 0$ for $x \notin A$. Similarly, since B is a G_δ set, $f_B(x) > 0$ for $x \notin B$.
Hence for $x \notin A \cup B$,

$$0 < \frac{f_A(x)}{f_A(x) + f_B(x)} < \frac{f_A(x)}{f_A(x) + 0} = 1$$

Therefore, $f^{-1}(0) = A$ and $f^{-1}(0) = B$ □